Bootstrapping of Discrete Barrier Options with Fourier Transforms

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Abstract

In this paper we propose an algorithm, based on the Fourier Transform technique, to "bootstrap" the term structure of discrete barrier options, from the volatility surface calibrated on European options. The technique provide a direct way to price barrier options on any arbitrary discrete tenor, with no need to resort to approximations from options with continuous time barriers. Application of the method is not limited to the standard Black and Scholes model, but extends to the class of Lévy processes proposed in recent years. We provide examples of applications to the Merton jump/diffusion model and the VG model. From comparison with high accuracy Monte Carlo it turns out that the main advantage of our method is speed: for comparable accuracy the Monte Carlo simulation is several orders of magnitude slower.

1 Introduction

Barrier options are among the most standard exotic products that may be found in the market for derivative instruments. In the standard literature on derivative products, one finds that a market for a product can completely specified by the spot price and the pricing kernel of European options, which are generally represented in terms of volatility surface. Actually, a complete description of the market should include pricing kernels for barrier options. This information would give the price of claims that pay a unit payoff if the underlying asset remain in a given region in a set of dates. Of course, there must be a link between European and barrier pricing kernels, which has to do with the requirement to rule out arbitrage possibilities. This argument leads to the static replication approach of Bowie and Carr (1994) and Carr, Ellis and Gupta (1998) for barrier options with continuous time monitoring. In the same spirit, Cherubini and Romagnoli (2009) proposed a "bootstrapping" algorithm to recover the term structure of barrier options with discrete monitoring from the corresponding term structure of digital options. In many instances, performing this bootstrapping application under completely loose assumptions about the stochastic process driving the underlying asset may be unpractical: in fact, liquid and transparent prices for all maturities and strikes are not available in almost any markets. The natural alternative is to impose more structure to the representation of the dynamics of the underlying asset, in order to fill the gaps in liquid quotes. In this paper we propose an efficient technique to accomplish this task.

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The analysis is performed under the assumption that the log-price of the underlying asset follows a Lévy process. The process is calibrated on the pricing kernel of European options and an algorithm based on the Fourier transform is proposed for the determination of a term structure of discrete barrier options. This way, we directly recover the whole term structure (under any arbitrary discrete tenor) consistent with the prices of European options without resorting to any approximation based on continuous barrier options (Broadie, Glasserman and Kou, 1997 correction, BGK since now on).
The application presented in this paper also highlights the usefulness of the Fourier transform technique not only to perform a parsimonious representation of the market by means of calibration (Duffie, Pan and Singleton, 2000, Carr and Madan 1999, Lewis 2001 and Cherubini et al., 2009), but also to provide information that would be hard to extract if we only had to rely the probability density of the price at a set of future dates. Moreover, such information can be extracted not only for the standard Black and Scholes setting on which most of the literature on discrete barrier options has focussed, but for a larger class of processes, namely those with independent and homogeneous increments.
To provide the main intuition behind the algorithm that we are going to propose, we may start from the argument that barrier options are characterized by the fact that the risk neutral probability must to be computed subject to some condition. The condition is expressed as the requirement that throughout the whole history the underlying must remain within some predefined domain or leave some domain. We will show that in many cases it is possible to compute this conditional probability density function (pdf) $p(S,T)$ performing a convolution in Fourier space. Once the pdf has been computed, we obtain with high accuracy the risk neutral price of the option performing directly the integral

$$\int dS\, p(S,T)\Phi(S),$$

where $\Phi(S)$ is the option payoff. We will show that beyond the standard Black and Scholes (BS) model, this method works well for most of the models currently in fashion, namely the Variance Gamma (VG) model, the jump diffusion model of Merton (M). From comparison of the results of the Fourier space convolution with high accuracy Monte Carlo simulations, it turns out that the main advantage of our method is speed: for comparable accuracy the Monte Carlo simulation is several orders of magnitude slower.

2 The model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{t_j\}_{j\geq 0}$ be a set of dates with $t_0 = 0$ and $t_j < t_{j+1}$. We assume a discrete time stochastic process $Y = \{Y_n\}_{n\geq 0}$ which is a random walk meaning

$$Y_n = \sum_{i=1}^{n} X_i, \qquad Y_0 = 0$$

with $\{X_i\}_{i\geq 1}$ independent random variables, and a discrete time stochastic process $S = \{S_n\}_{n\geq 0}$ defined as

$$S_n = S_0 e^{Y_n},$$

where, by sake of simplicity, we set

$$X_n = X(t_n), \quad Y_n = Y(t_n), \quad S_n = S(t_n).$$

If, for $n \geq 1$ $\Omega_n \subset \mathbb{R}$ is a Borel set, let us consider the events

$$\varphi_n \equiv \{Y_n \in \Omega_n\}$$

and

$$\Phi_n = \bigcap_{i=1}^{n} \varphi_i.$$

For every Borel set $A \subset \mathbb{R}$, let, for $m \leq n$,

$$\mu_{n,m}(A) = \mathbb{P}(\{Y_n \in A\} \cap \Phi_m).$$

We assume the measure $\mu_{n,m}$ to have a density $p_Y(y, t_n, \Phi_m)$ so that

$$\mu_{n,m}(A) = \int_A dy\, p_Y(y, t_n, \Phi_m). \tag{1}$$

Notice that

$$\int_A dy\, p_Y(y, t_n, \Phi_n) = \mu_{n,n}(A) = \mathbb{P}(\{Y_n \in A \cap \Omega_n\} \cap \Phi_{n-1}) =$$
$$= \int_{A\cap\Omega_n} dy\, p_Y(y, t_n, \Phi_{n-1}) = \int_A dy\, p_Y(y, t_n, \Phi_n)\mathbf{1}_{\Omega_n}$$

from which

$$p_Y(y, t_n, \Phi_n) = p_Y(y, t_n, \Phi_{n-1})\mathbf{1}_{\Omega_n}.$$

If the distribution of X_i, $\mathbb{P}_{X_i}(A) = \mathbb{P}_X(A, t_i - t_{i-1})$, admits a density $p_{X_i}(x) = p_X(x, t_i - t_{i-1})$, that is

$$\mathbb{P}_{X_i}(A) = \int_\eta dx\, p_{X_i}(x),$$

then, such a density gives the transition density of the process. More precisely, the transition density $T(x_1, t_i; x_2, t_{i+1})$ from the state x_1 at t_i to the state x_2 at t_{i+1} is a function of the differences $(x_2 - x_1)$ and $(t_{i+1} - t_i)$ and

$$T(x_1, t_i; x_2, t_{i+1}) = p_X(x_2 - x_1, t_{i+1} - t_i).$$

If for every Borel set A and $c \in \mathbb{R}$ we define $A^c = \{w \in \mathbb{R} : w = y - c \text{ with } y \in A\}$, then

$$\mu_{n,m}(A) = \int_{\mathbb{R}} dz \mathbb{P}\left(\{Y_{n-1} + z \in A\} \cap \Phi_m \middle| X_n = z\right) p_X(z, t_n - t_{n-1}) =$$
$$= \int_{\mathbb{R}} dz \mathbb{P}\left(\{Y_{n-1} \in A^z\} \cap \Phi_m\right) p_X(z, t_n - t_{n-1}) =$$
$$= \int_{\mathbb{R}} dz \mu_{n-1,m}(A^z)\, p_X(z, t_n - t_{n-1}) =$$
$$= \int_{\mathbb{R}} dz \int_{A^z} dx\, p_Y(x, t_{n-1}, \Phi_m) p_X(z, t_n - t_{n-1}) =$$
$$= \int_{\mathbb{R}} dx \int_A dy\, p_X(y - x, t_n - t_{n-1}) p_Y(x, t_{n-1}, \Phi_m).$$

Hence

$$\mu_{n,m}(A, \Phi_m) = \int_A dy \int_{\mathbb{R}} dx\, p_X(y - x, t_n - t_{n-1}) p_Y(x, t_{n-1}, \Phi_m)$$

and by (1)

$$p_Y(y, t_n, \Phi_m) = \int_{\mathbb{R}} dx\, p_X(y - x, t_n - t_{n-1}) p_Y(x, t_{n-1}, \Phi_m).$$

In other words, $\mu_{n,m}$ is the convolution of $\mathbb{P}_X$ and $\mu_{n-1,m}$, that is

$$\mu_{n,m} = \mathbb{P}_{X_n} * \mu_{n-1,m}.$$

Given a measure μ with density p its Fourier transform (or, equivalently, the Fourier transform of p) is defined as

$$\hat{\mu}(k) = \mathcal{F}\mu(k) = \int e^{2\pi kz} p(z)dz,$$

while the inverse Fourier transform is defined as

$$p(z) = \overline{\mathcal{F}}\hat{\mu}(z) = \int e^{-2\pi kz} \hat{\mu}(k)dk.$$

It is well known that the Fourier transform of a convolution is the product of the corresponding Fourier transforms, hence

$$\hat{\mu}_{n,m}(k) = \hat{\mathbb{P}}_{X_n}(k)\hat{\mu}_{n-1,m}(k).$$

Given these results, we can recover the sequence of densities $p_Y(x, t_n, \Phi_n)$ by iteratively applying the following algorithm:

The Fourier Transform convolution algorithm.

1. $\hat{\mu}_{n,n}(k) = \mathcal{F}\mu_{n,n}(k)$
2. $\hat{f}_n(k) = \hat{\mathbb{P}}_{X_n}(k)\hat{\mu}_{n,n}(k)$
3. $p_Y(x, t_{n+1}, \Phi_n) = \overline{\mathcal{F}}\hat{f}_n(x)$
4. $p_Y(x, t_{n+1}, \Phi_{n+1}) = p_Y(x, t_{n+1}, \Phi_n)\mathbf{1}_{\Omega_n}$

3 Discrete Barrier Options

Given a set of fixing date $t_1, \ldots, t_n$. between 0 and T, lets consider a digital option $\mathcal{Y}(T)$ written on an underlying S_t that pays at T a unit amount if the underlying S_t satisfies a given constraint at every fixing date, that is, in line with the formalism of the previous section we shall express these events as

$$\varphi_n \equiv \{S(t_n) \in \Sigma_n\}\,.$$

A very practical situation is when the set Σ_n is an interval of the real line, defined by an upper barrier B_h and a lower barrier B_l. In this case, the condition φ_n is expressed in terms of the process $Y(t_n)$ as:

$$\varphi_n = \left\{ \log\left(\frac{B_l}{S_0}\right) < Y(t_n) < \log\left(\frac{B_h}{S_0}\right) \right\}$$

and, in line with the symbology of the previous section $\Omega_n = \left(\log\left(\frac{B_l}{S_0}\right), \log\left(\frac{B_h}{S_0}\right)\right)$. Therefore, using the FT convolution algorithm of the previous section, once we have computed $p_Y(x, t_n, \Phi_{n-1})$, $p_Y(x, t_n, \Phi_n)$ is given by:

$$p_Y(x, t_n, \Phi_n) = \begin{cases} 0 & x \leq \log(B_l/S_0) \\ p_Y(x, t_n, \Phi_{n-1}) & \log(B_l/S_0) < x < \log(B_h/S_0) \\ 0 & x \geq \log(B_h/S_0) \end{cases}$$

3.1 Implementation details

We work with the process

$$\overline{S}_t = S_0 e^{X_t}$$

that is a martingale with respect to the risk neutral measure, meaning that if S_t is the standard risk neutral process for an asset, $\overline{S}_t = S_t/B(t)$ where $B(t)$ is the bank account process. The processes X_t that we consider are processes where we can easily compute the characteristic function and are spatially homogeneous, meaning that the transition probability $p(X_{t_{n+1}} \leftarrow X_{t_n})$ from X_{t_n} to $X_{t_{n+1}}$ depends only on the distance $X_{t_{n+1}} - X_{t_n}$. This means, for instance, that we can use Levy processes but not the Heston process. Among Levy process we will work with the standard Black-Scholes model, the Variance Gamma model and the jump-diffusion Merton model.
The transition function can be seen (actually is) the conditional probability to be in $X_{t_{n+1}}$ at $t = t_{n+1}$ given the constraint to be in X_{t_n} at $t = t_n$. With a slight abuse of notation, and a significant amount os sloppiness, we can write:

$$p(X_{t_{n+1}} \leftarrow X_{t_n}) = P(X_{t_{n+1}}, t_{n+1} | X_{t_n}, t_n)$$

and from the spatial homogeneity we can conclude that

$$p(X_{t_{n+1}} \leftarrow X_{t_n}) = P_n(X_{t_{n+1}} - X_{t_n}, t_{n+1} - t_n),$$

furthermore, since we confine ourselves to Levy processes, we can write:

$$P_n(X_{t_{n+1}} - X_{t_n}, t_{n+1} - t_n) = P(X_{t_{n+1}} - X_{t_n}, t_{n+1} - t_n)$$

showing that the needed transition probability is nothing but the unconditional probability distribution for the process $X_{t_{n+1}} - X_{t_n}$ and this can easily be obtained from the characteristic function of the process.

Model	Characteristic Function	Parameters
Black-Scholes	$\exp\left(-2\pi i u \frac{\sigma^2}{2}\Delta - \frac{(2\pi u\sigma)^2\Delta}{2}\right)$	σ:volatility
Merton	$\exp\left[\left(-\pi u\sigma^2\,(i + 2\pi u) - \lambda(1 - \phi_J(u)) + 2\pi i u\lambda\left(1 - \phi_J\left(\frac{1}{2\pi i}\right)\right)\right)\Delta\right]$	σ : volatility λ : jump intensity ϕ_J : c.f. of jumps
Variance-Gamma	$\exp\left(\frac{i2\pi ut}{\nu}\log\left(1 - \nu\theta - \frac{\nu\sigma^2}{2}\right)\right)\left(\frac{1}{1+2\nu(\pi k\sigma)^2 - i2\pi k\nu\theta}\right)^{t/\nu}$	σ : volatility θ : skewness ν : kurtosis

From a practical application point of view, what needs to be kept in mind is that the formalism of the previous section is going to produce for us the conditional probability for the process

$$X_t = \frac{\overline{S}_t}{S_0}$$

but the barrier condition, say a high barrier, has to be imposed on the process $S_t = B(t)\overline{S}_t$, that is

$$X_t < \log\left(\frac{B_h}{B(t)S_0}\right)$$

that amounts to rescaling the barrier B_h the forward price $F^w(t) = S_0 B(t)$.

4 Numerical Tests

We now use the procedure described above to compute, rather efficiently, the effect of discrete fixing for options with barriers. Let us formalize the problem by selecting a set of fixing dates

$$0 = t_0, t_1, \ldots, t_N = T$$

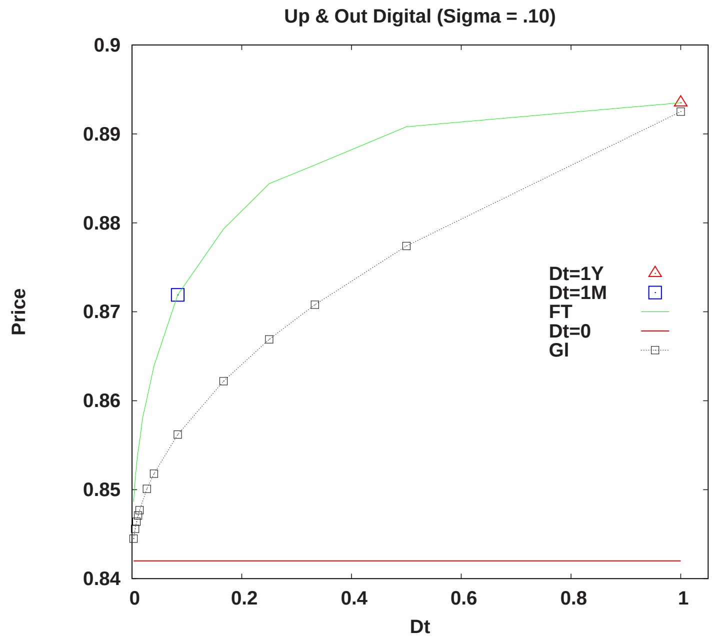

Figure 1: Convergence to the continuum limit for the Up and Out digital Option. $\sigma = 0.10, \quad r = .04$ The price is plotted vs. the fixing interval. The curve labelled "Gl" is the BGK correction to the continuum limit.

At T the option expires and the agreed payoff is paid if the condition

$$\varphi \equiv \bigcap_{i=1}^{N} (S(t_i) < B_h)$$

is met.
The simplest payoff we can consider is digital

$$\mathcal{C}_n = \begin{cases} 1 & \varphi \text{is true} \\ 0 & \varphi \text{is false} \end{cases},$$

and the payment is due if and only if at every fixing the underlying is below the barrier.
The price of such an option is given by:

$$\Pi(0, \mathcal{C}) = e^{-rT} \mathbb{E}[\, \mathbf{1}_{[\varphi]} \,]$$

There are two limits where this "*Up and Out*" digital can be computed in closed form, at least for the Black-Scholes model, that is $N = 1$ and $N = \infty$. The case $N = 1$ reduces to a standard *Cash or Nothing* with strike equal to the barrier, while the case $N = \infty$ covers the case of a barrier Up and Out with continuous time fixing, also known as *No-Touch*.
In the Black and Scholes setting, the analytical result for the case $N = 1$ is given by:

$$\Pi(0, \mathcal{C}; N = 1) = e^{-rT} N \left(\frac{\log(B_h/S_0) - \mu T}{\sqrt{\sigma^2 T}} + \frac{\sigma \sqrt{T}}{2} \right)$$

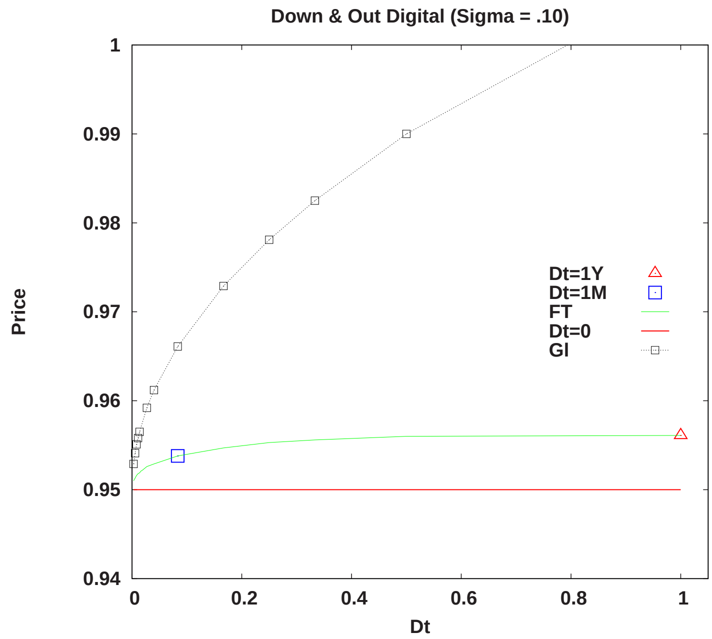

Figure 2: Convergence to the continuum limit for the Down and Out digital Option. $\sigma = 0.10, \quad r = .04$ The price is plotted vs. the fixing interval. The curve labelled "Gl" is the BGK correction to the continuum limit.

while for the case $N = \infty$ is:

$$\begin{aligned}\Pi(0, \mathcal{C}; N = \infty) &= N\left(\frac{\log(B_h/S_0) - \mu T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) \\ &\quad - \left(\frac{B_h}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1}\left[1 - N\left(\frac{\log(B_h/S_0) + \mu T}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)\right]\end{aligned}$$

In fig. (**??**) we show the transition from the $N = 1$ to $N = \infty$ result as a function of the fixing interval for a relatively low volatility, while the same plot is made in fig. (**??**) for the *Down-and-Out* option. In figs. (**??**, **??**) the same transition is shown as a function of the number of fixings. In all cases we study options expiring in one year. The results for intermediate fixings have been obtained by the FT-convolution method and they agree extremely well with analogous results obtained via Monte Carlo simulation. The purpose of these plots is twofold. One one hand we use it to confirm the accuracy of the FT-convolution method that correctly reproduces MC results, on the other we would like to warn against relying too much on the analytical results for $N = \infty$ anytime we are faced with pricing barrier options by simulation.

5 Results

In all of the numerical results we list below we have compared the numerical results of the FT conditional distribution with a standard MC method. In all cases we have taken into considerations three models: the Black-Scholes (BS) model, the Variance Gamma (VG)

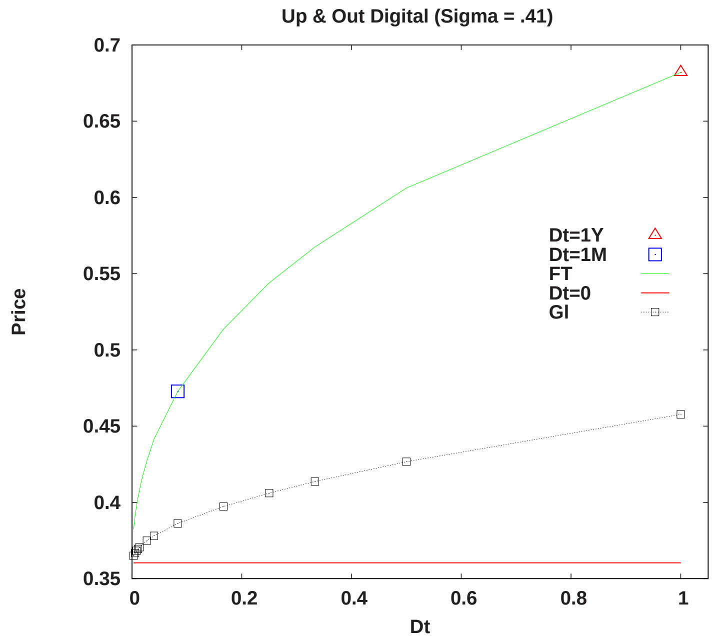

Figure 3: Convergence to the continuum limit for the Up and Out digital Option. $\sigma = 0.41$, $r = .04$ The price is plotted vs. the fixing interval. The curve labelled "Gl" is the BGK correction to the continuum limit.

model and the jump-diffusion Merton (M) model. Computing times are not listed but the over performance of the FT conditional distribution method is always above three orders of magnitude.
The quoted error for the MC results were obtained by running 10^{20} iterations. The VG model and the Merton model were calibrated to the volatility surface shown in fig. (5), while for the BS model we used the ATM implied volatility for the six months period. The parameters for the VG model are: $\theta = -0.5114$, $\nu = 0.03286$, $\sigma = 0.4450$; for the Merton model are: $\lambda = 0.4931$, $\mu = -0.041$, $\eta = 0.041$ and $\sigma = 0.4406$ while for the BS model we have $\sigma = 0.4100$. We are now going to show how to recover the price of barrier options from the pricing kernels we computed.

5.1 Channel Options

The first payoff we look at is a channel option, that is an option that pays the agreed amount if and only if the underlying remains, throughout the life of the option, within a low barrier B_l and a high barrier B_h.
To make the statement more formal we define the event

$$\phi = [\, B_l < \underline{S} \cap \overline{S} < B_h; \quad \underline{S} = \min(S(t), 0 \le t \le T); \quad \overline{S} = \max(S(t), 0 \le t \le T)\,]$$

and we look at the price of cash or nothing payoff

$$\mathcal{C}_n = 1, \quad \Pi(0, \mathcal{C}_n) = \mathbb{E}[D(0,T)\mathbf{1}_{[\phi]}\,]$$

and the asset or nothing

$$\mathcal{A}_n = S(T), \quad \Pi(0, \mathcal{A}_n) = \mathbb{E}[D(0,T)S(T)\mathbf{1}_{[\phi]}\,]$$

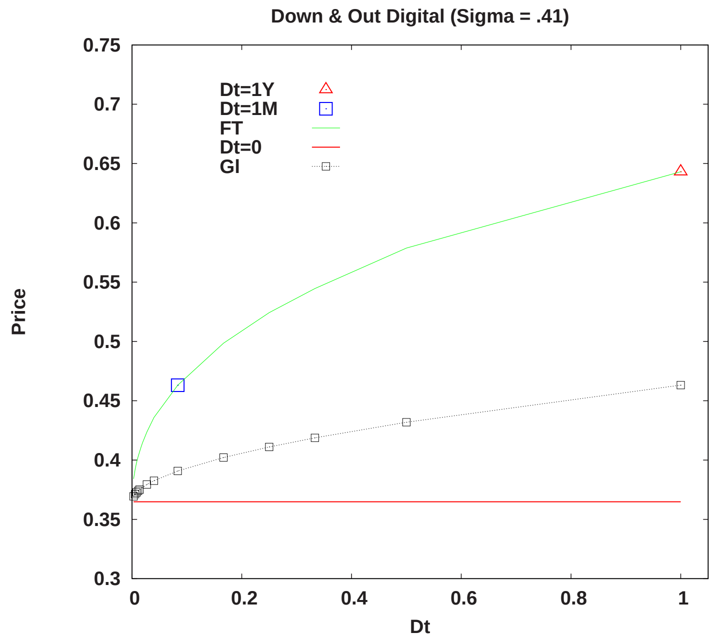

Figure 4: Convergence to the continuum limit for the Down and Out digital Option. $\sigma = 0.41, \quad r = .04$ The price is plotted vs. the fixing interval. The curve labelled "Gl" is the BGK correction to the continuum limit.

In table (1) we compare the standard MC method with the FT method for the three models described above for a one year living option and fixing every month. While in table (**??**) we compare the same thing except that the fixing interval is three months.

Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$\mathbb{E}[D(0,T)S_T\mathbf{1}_{h[\phi]}]$	0.261	0.260 ± 0.001	0.221	0.220 ± 0.001	0.223	0.222 ± 0.001
$\mathbb{E}[D(0,T)\mathbf{1}_{h[\phi]}]$	0.304	0.303 ± 0.001	0.256	0.255 ± 0.001	0.260	0.259 ± 0.001

Table 1: . "Cash or Nothing" and "Asset or Nothing" channel options. The fixing period is one month, $T = 1Y$

5.2 Down and Out Call, Down and In Put

The next payoff we consider deals only with the low barrier. We look at a "down and out" call option. This is an option subject to the condition to become worthless if at some fixing the underlying touches or goes below the lower barrier. Symmetrically we look at a "down and in" put option: an option that remains worthless if the underlying never goes below a low barrier.
Let's define the statement:

$$\varphi = [\ B_l < \underline{S}, \quad \underline{S} = \min(S(t), 0 \le t \le T);\ ]$$

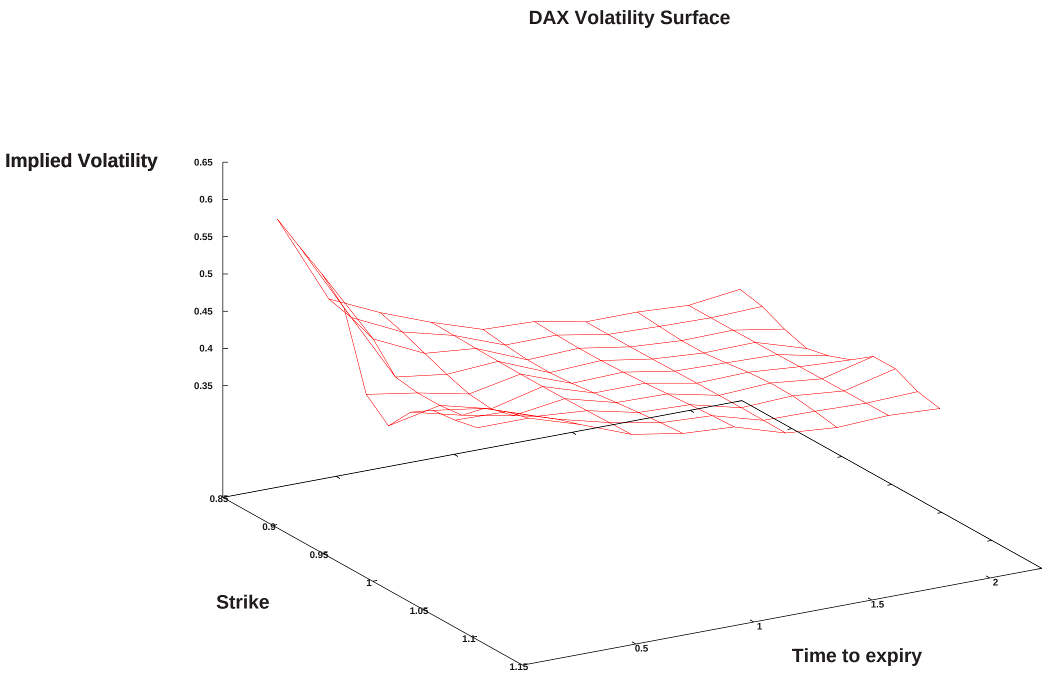

Figure 5: The volatility surface we have been working with.

Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$\mathbb{E}[D(0,T)S_T\mathbf{1}_{h[\phi]}]$	0.342	0.340 ± 0.001	0.298	0.297 ± 0.001	0.306	0.305 ± 0.001
$\mathbb{E}[D(0,T)\mathbf{1}_{h[\phi]}]$	0.395	0.393 ± 0.001	0.344	0.342 ± 0.001	0.355	0.354 ± 0.001

Table 2: . "Cash or Nothing" and "Asset or Nothing" channel options. The fixing period is three months, $T = 1Y$

the price of the "down and out call" is given by:

$$\Pi(0,\mathcal{C}) = \mathbb{E}[\,D(0,T)[S(T)-K]^+\mathbf{1}_{h[\varphi]}\,]$$

while the "down and in" put is given by:

$$\Pi(0,\mathcal{P}) = \mathbb{E}[\,D(0,T)[K-S(T)]^+\mathbf{1}_{h[\varphi^c]}\,]$$

5.3 Up and In Call, Up and Out Put

At last the payoff we consider deals only with the high barrier. We look at a "up and in" call option: an option subject to the condition that to remain worthless if underlying never crosses the high barrier. Symmetrically we look at a "up and out" put option: an option that becomes worthless if the underlying crosses the high barrier.
Let's define the statement:

$$\psi = [\,\overline{S} < B_h; \quad \overline{S} = \max(S(t), 0 \le t \le T)\,]$$

Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$[S_T - K]^+ \mathbf{1}_{h[\varphi]}$	0.165	0.165 ± 0.001	0.176	0.177 ± 0.001	0.173	0.173 ± 0.001
$[K - S(T)]^+ \mathbf{1}_{h[\varphi^c]}$	0.132	0.132 ± 0.001	0.148	0.148 ± 0.001	0.145	0.145 ± 0.001

Table 3: . "Down and Out Call" and "Down and In Put" options. The fixing period is one months, $T = 1Y$, $B_l = .8$, $K = 1.0$

Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$[S_T - K]^+ \mathbf{1}_{h[\varphi]}$	0.171	0.172 ± 0.001	0.184	0.184 ± 0.001	0.181	0.181 ± 0.001
$[K - S(T)]^+ \mathbf{1}_{h[\varphi^c]}$	0.128	0.128 ± 0.001	0.144	0.144 ± 0.001	0.142	0.142 ± 0.001

Table 4: . "Down and Out Call" and "Down and In Put" options. The fixing period is three months, $T = 1Y$, $B_l = .8$, $K = 1.0$

the price of the "up and in call" is given by:

$$\Pi(0, \mathcal{C}) = \mathbb{E}[\, D(0,T)[S(T) - K]^+ \mathbf{1}_{h[\psi^c]} \,]$$

while the "up and out" put is given by:

$$\Pi(0, \mathcal{P}) = \mathbb{E}[\, D(0,T)[K - S(T)]^+ \mathbf{1}_{h[\psi]} \,]$$

Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$[S_T - K]^+ \mathbf{1}_{h[\psi^c]}$	0.175	0.175 ± 0.001	0.189	0.190 ± 0.001	0.188	0.187 ± 0.001
$[K - S(T)]^+ \mathbf{1}_{h[\psi]}$	0.121	0.121 ± 0.001	0.130	0.130 ± 0.001	0.129	0.128 ± 0.001

Table 5: . "Up and In Call" and "Up and Out Put" options. The fixing period is one months, $T = 1Y$, $B_h = 1.2$, $K = 1.0$

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Payoff	BS FT	BS MC	VG-FT	VG-MC	M-FT	M-MC
$[S_T - K]^+ \mathbf{1}_{h[\psi^c]}$	0.172	0.172 ± 0.001	0.187	0.187 ± 0.001	0.185	0.185 ± 0.001
$[K - S(T)]^+ \mathbf{1}_{h[\psi]}$	0.129	0.129 ± 0.001	0.140	0.140 ± 0.001	0.139	0.139 ± 0.001

Table 6: . "Up and In Call" and "Up and Out Put" options. The fixing period is three months, $T = 1Y$, $B_h = 1.2$, $K = 1.0$

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