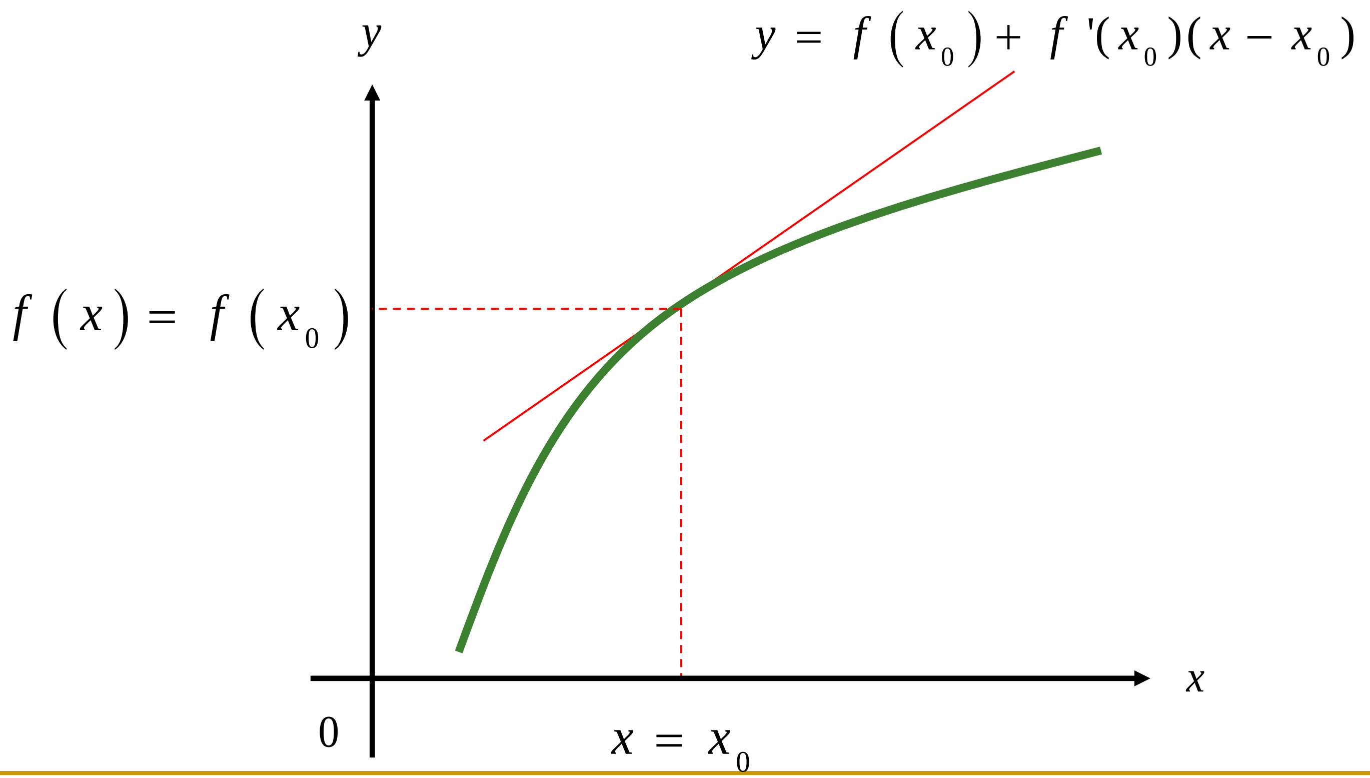
Università degli Studi di Siena

Electromagnetism

Mathematical Interlude: Taylor series

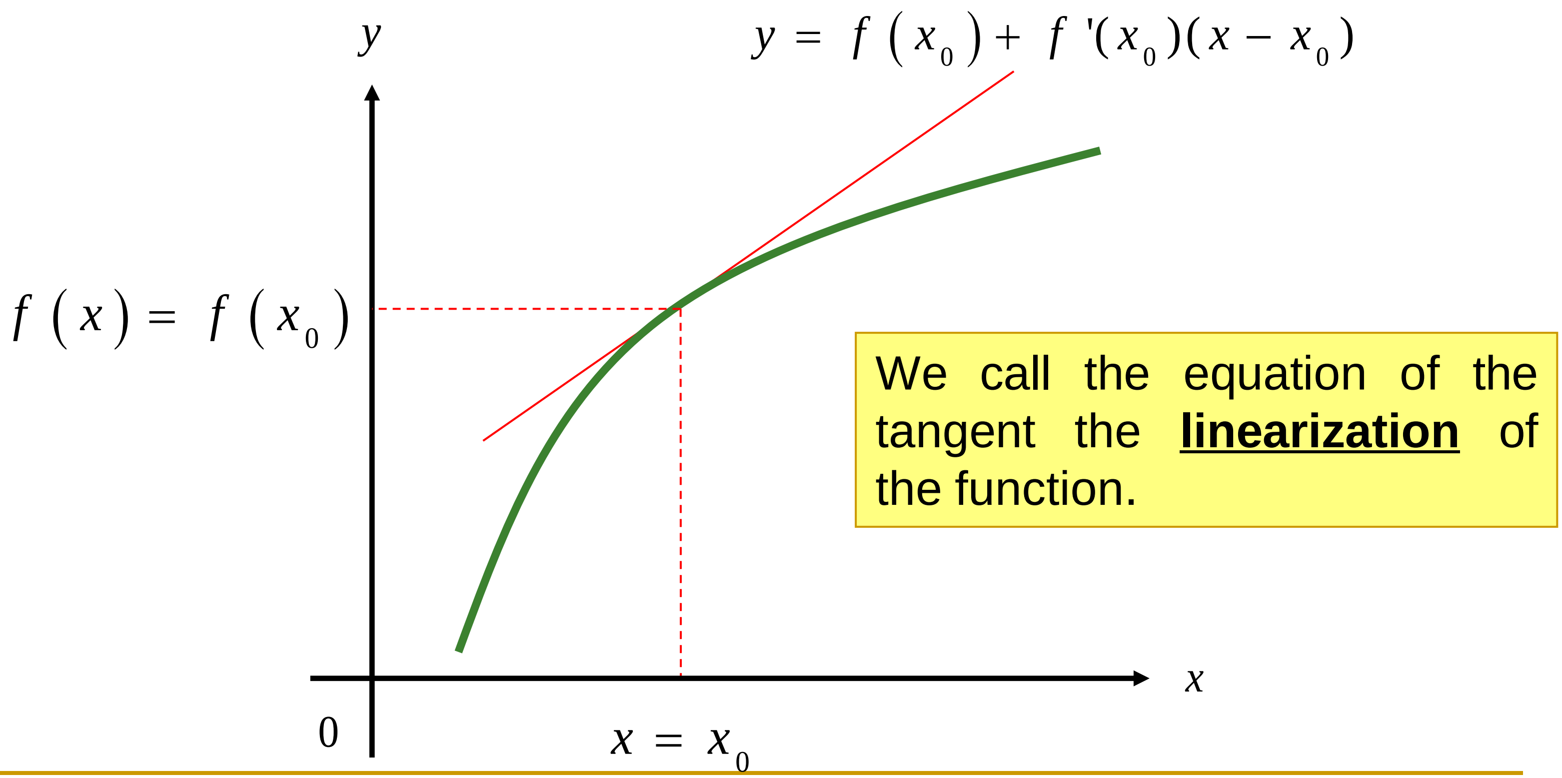
Taylor Series

- In mathematics, a Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.
- Recall: the geometrical idea of the tangent line of a differentiable function



Taylor Series

- For a function $f(x)$ that is differentiable at $x = x_0$, the tangent is a close approximation of the function in a neighborhood of the tangent point x_0.

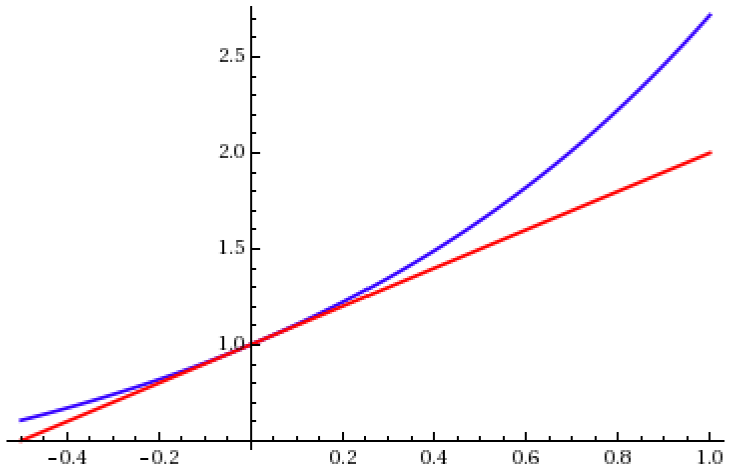


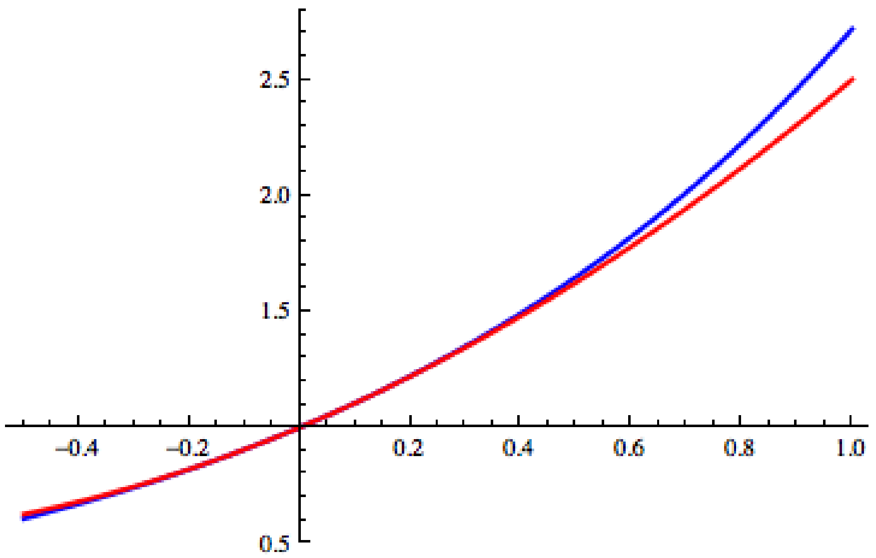
Ambition:

- Approximating of differential functions more precisely by polynomials of higher degrees.
- If we wanted a better approximation to f, we might instead try a quadratic polynomial instead of a linear function. Instead of just matching one derivative of f at a, we can match two derivatives, thus producing a polynomial that has the same slope and concavity as f at a. The quadratic polynomial in question is

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Ambition:





Taylor Polynomials

- Similarly, we get still better approximations to f if we use polynomials of higher degree, since then we can match even more derivatives with f at the selected base point. In general, the error in approximating a function by a polynomial of degree k will go to zero a little bit faster than $(x-a)^k$ as x tends to a.
- The Taylor polynomial of degree n for a function $f(x)$ which is n times continuously differentiable at x_0 is

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$
$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Taylor Polynomials

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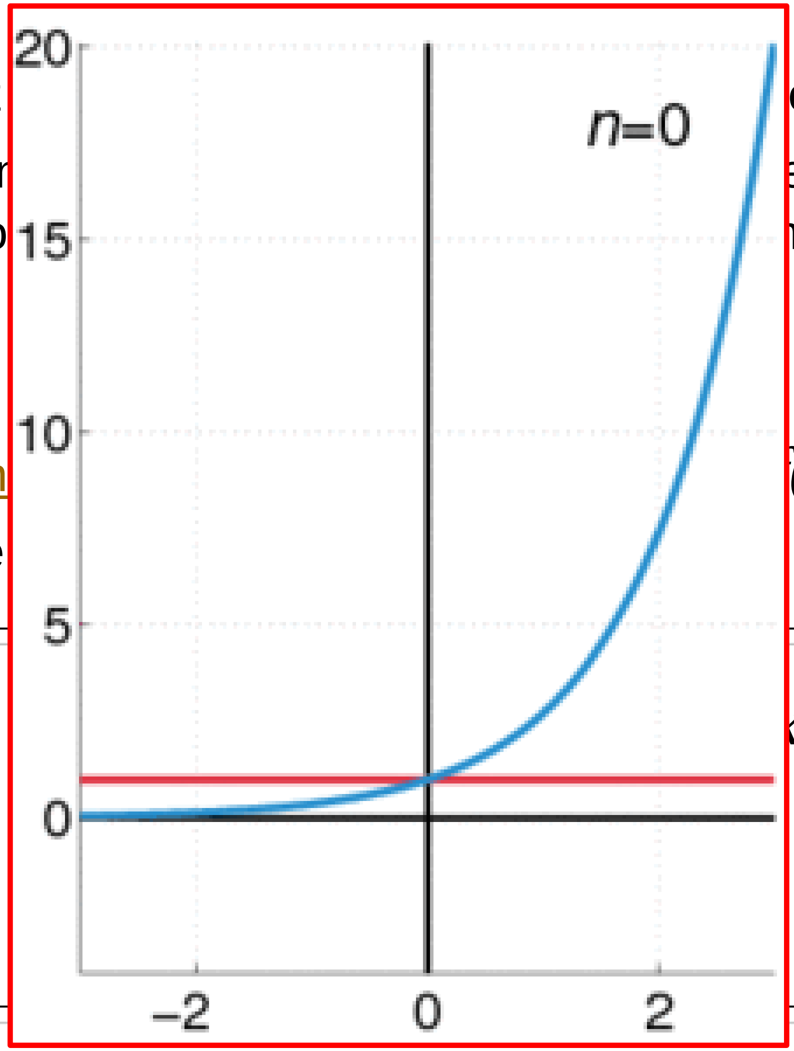
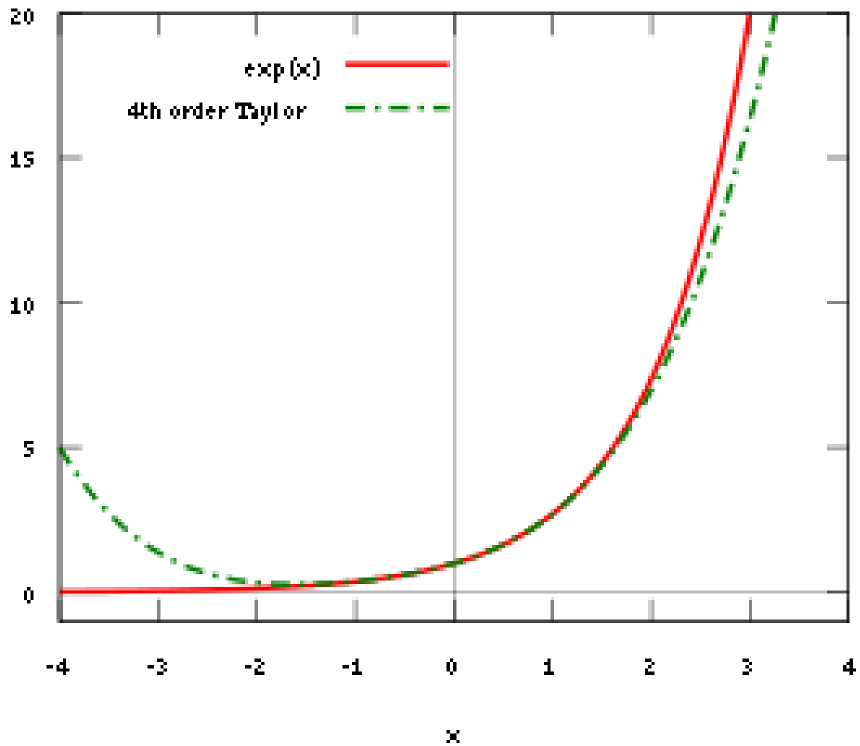


$$f(x_0) + \ldots (x - x_0)^2 + \ldots$$

Example:

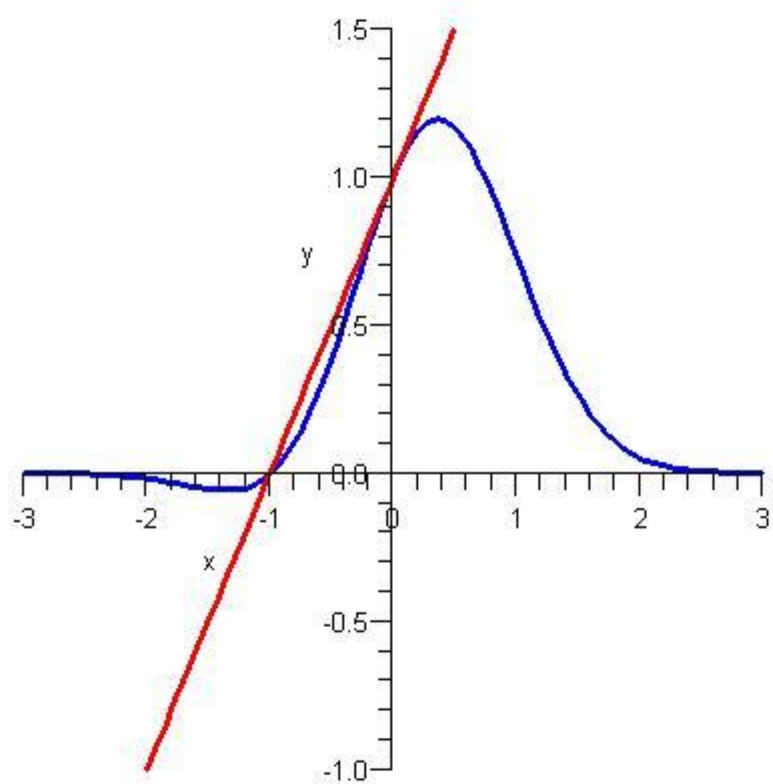
computing $f(x) = e^x$ using Taylor Series



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

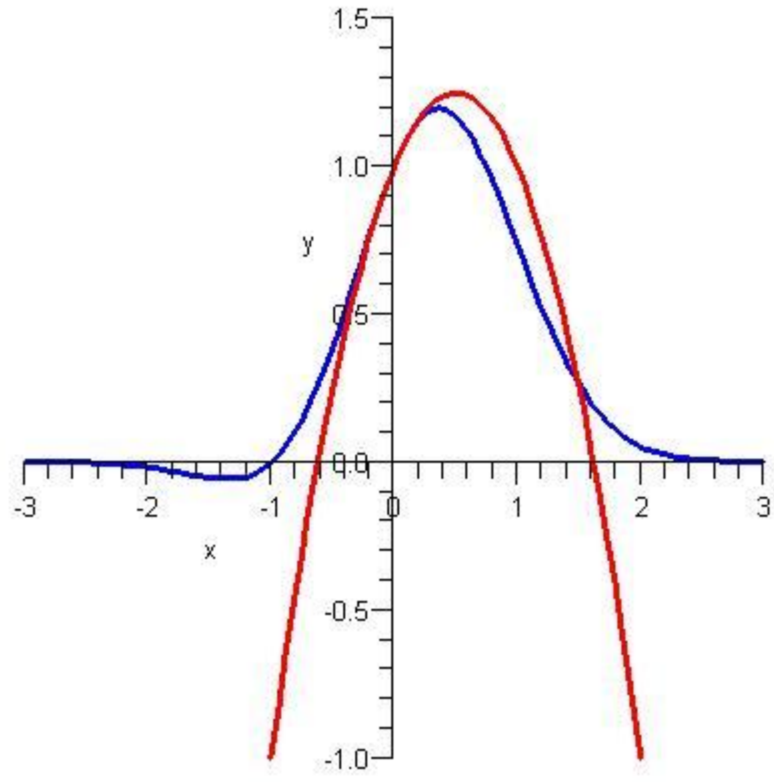
Approximating $f(x) = (x+1)e^{-x^2}$

- Best first order (linear) approximation at $x=0$.
- we can call this straight line function $P_1(x)$.
- Note: $f(0)=P_1(0)$ and $f'(0) = P'_1(0)$.



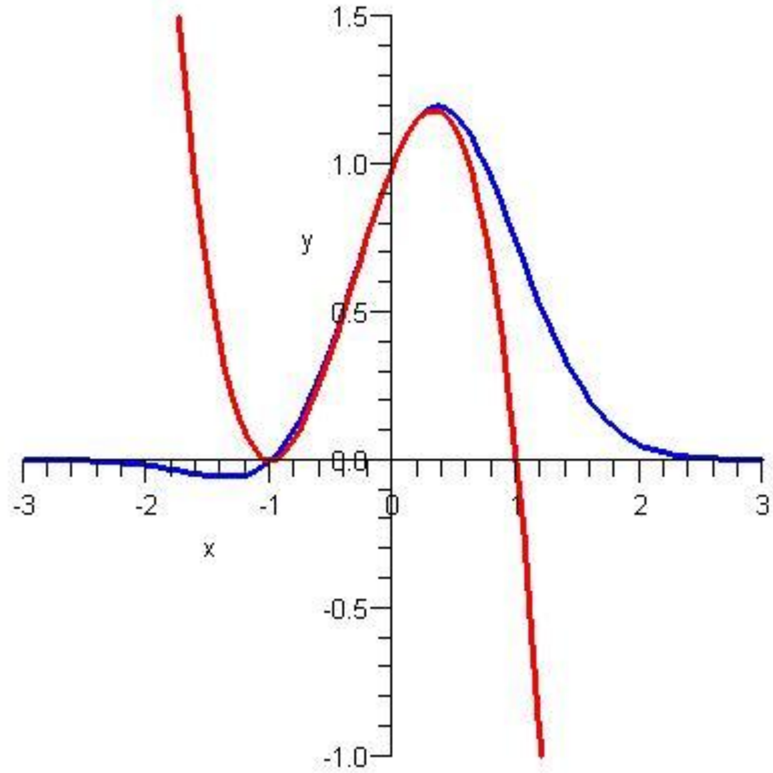
Approximating $f(x) = (x+1)e^{-x^2}$

- Best second order (quadratic) approximation at $x=0$.
- We can call this quadratic function $P_2(x)$.
- Note: $f(0)=P_2(0)$, $f'(0) = P'_2(0)$, and $f''(0) = P''_2(0)$.



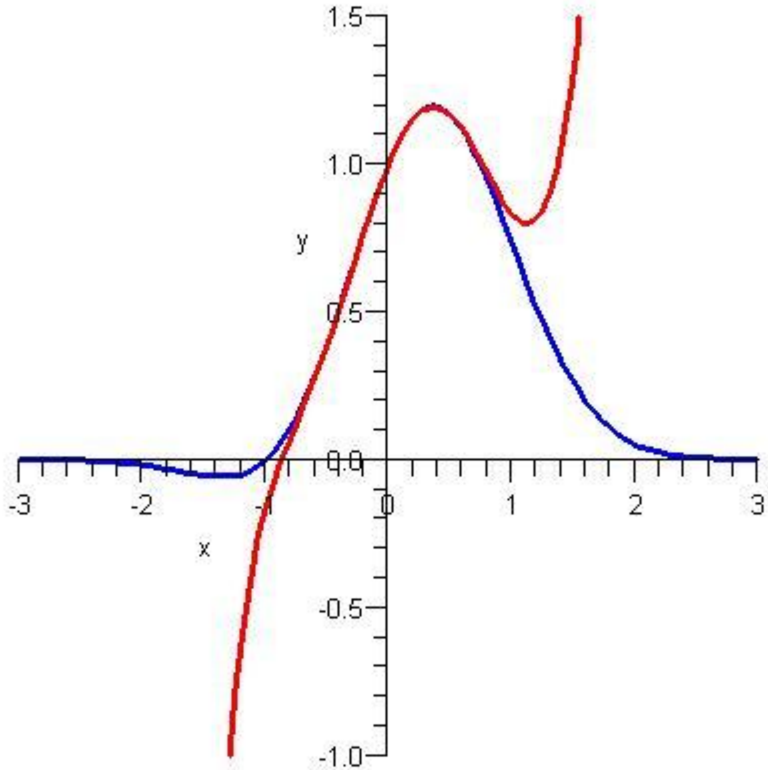
Approximating $f(x) = (x+1)e^{-x^2}$

- Best third order (cubic) approximation at $x=0$.
- We can call this cubic function $P_3(x)$.
- Note: $f(0)=P_3(0)$,
- $f'(0) = P'_3(0)$,
  $f'(0) = P''_3(0)$, and
  $f'''(0) = P'''_3(0)$.



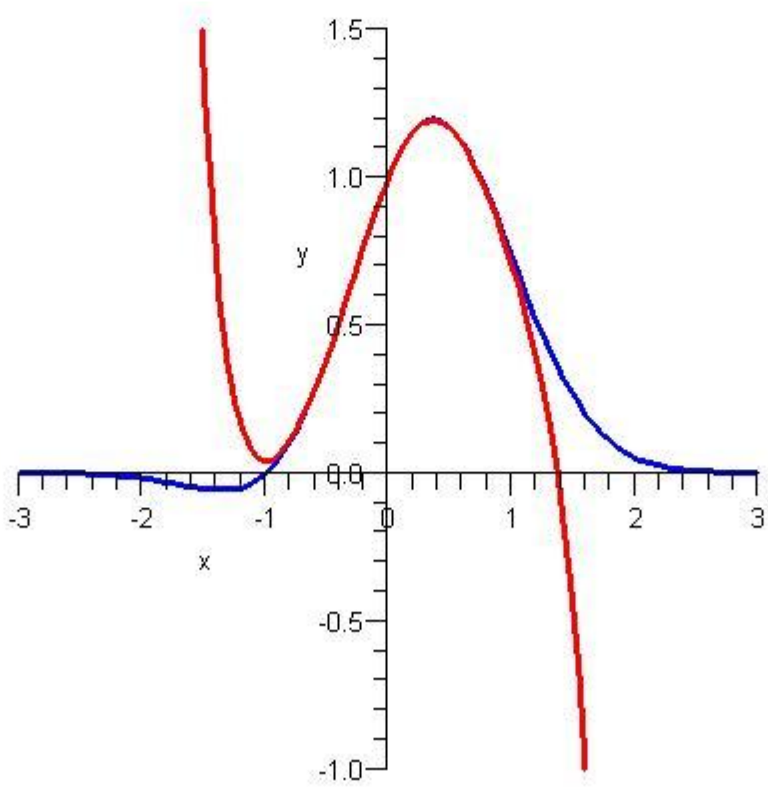
Approximating $f(x) = (x+1)e^{-x^2}$

- Best sixth order approximation at $x=0$.
- We can call this function $P_6(x)$.
- P_6 "matches" the value of f and its first six derivatives at $x = 0$.



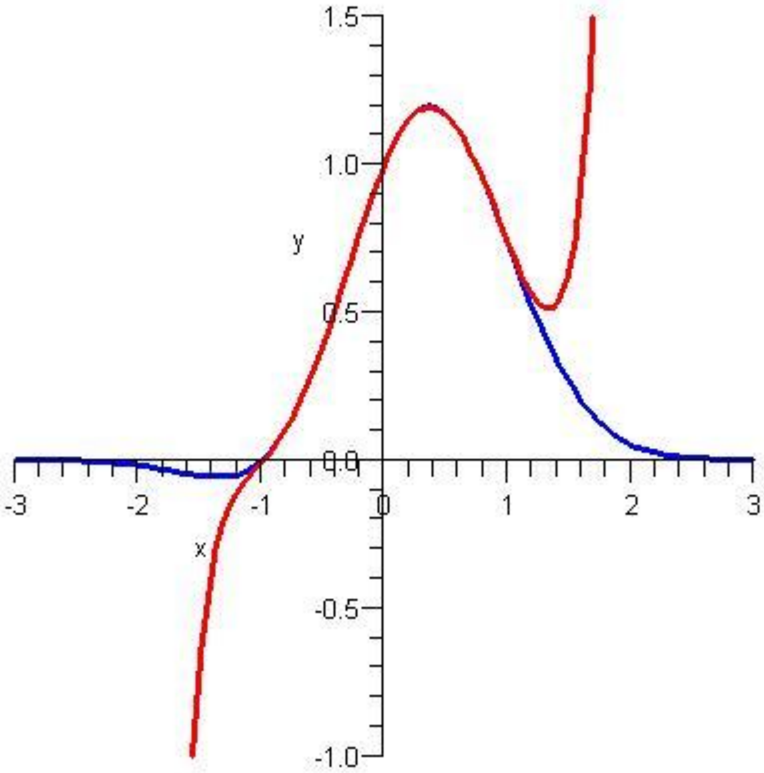
Approximating $f(x) = (x+1)\,e^{-x^2}$

- Best eighth order approximation at $x=0$.
- We can call this function $P_8(x)$.
- P_8 "matches" the value of f and its first eight derivatives at $x = 0$.



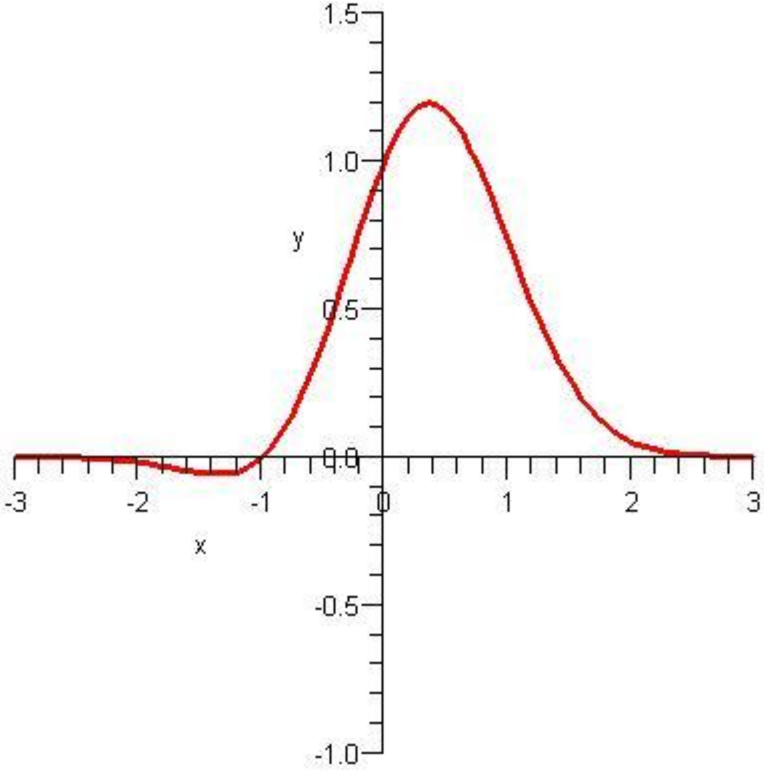
Approximating $f(x) = (x+1)e^{-x^2}$

- Best tenth order approximation at $x=0$.
- This function is $P_{10}(x)$.



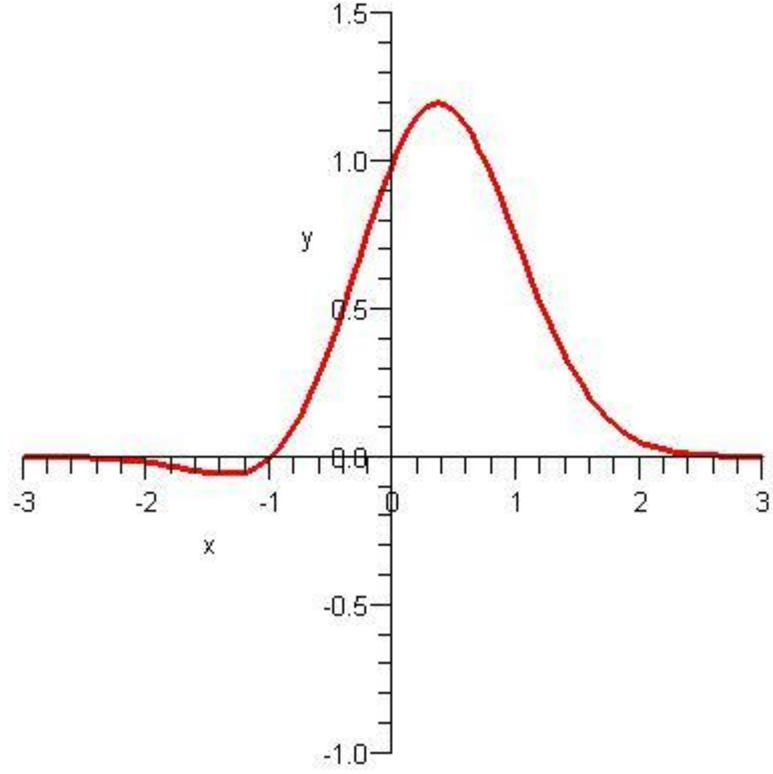
Approximating $f(x) = (x+1)e^{-x^2}$

- Best hundredth order approximation at $x=0$.
- This function is $P_{100}(x)$.
- Notice that we cannot see any difference between f and P_{100} on the interval [-3,3].



Approximating $f(x)=(x+1)\,e^{-x^2}$

- Best hundredth order approximation at $x=0$.
- This function is $P_{100}(x)$.
- But what about [-6,6]?



Approximating $f(x) = (x+1)\,e^{-x^2}$

- Best hundredth order approximation at $x=0$.
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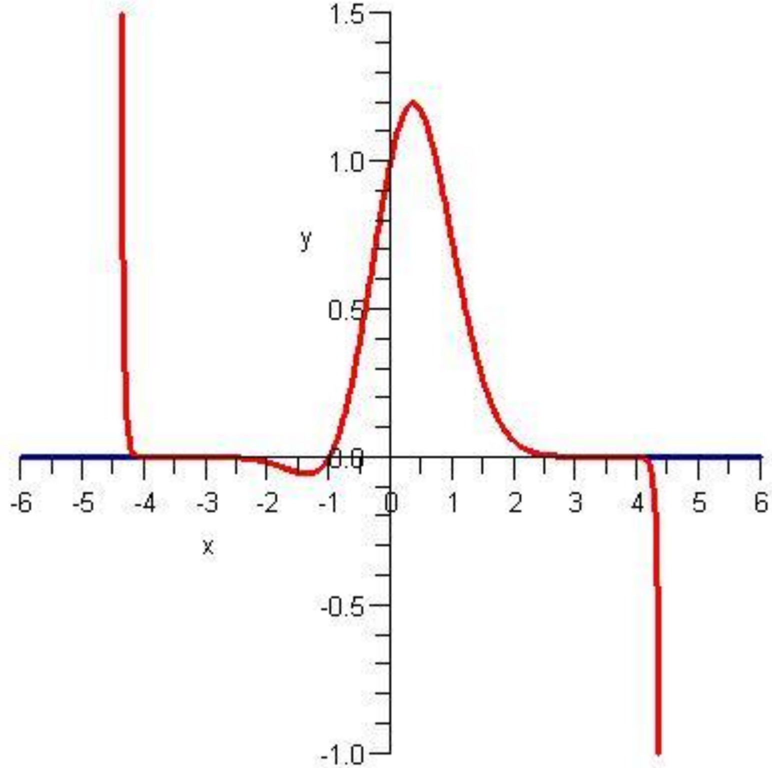
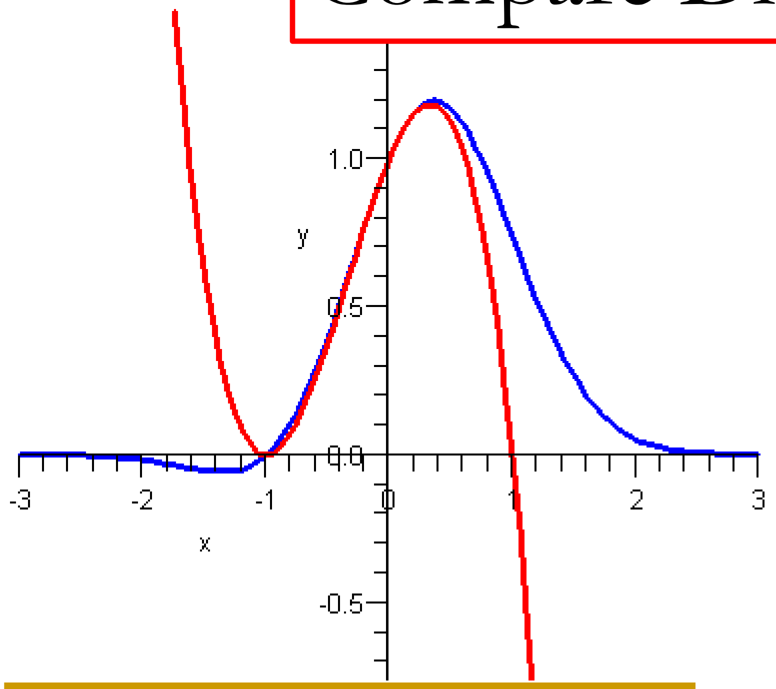
- But what about [-6,6]?

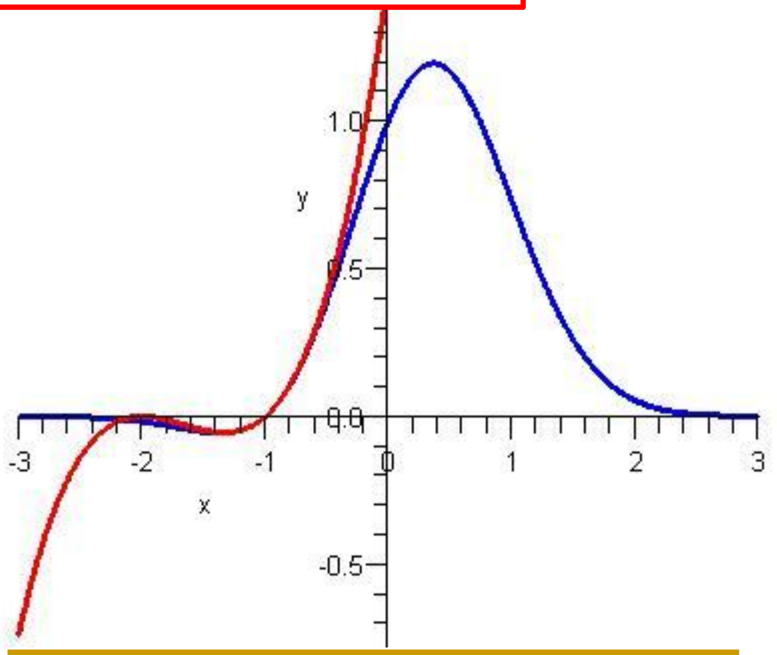


Approximating $f(x) = (x+1)e^{-x^2}$

Compare Different Centers



Third order approximation at $x=0$



Third order approximation at $x = -1$