A Copula-based Model of Speculative Price Dynamics in Discrete Time

Umberto Cherubini * Sabrina Mulinacci†

Silvia Romagnoli ‡

May 31, 2009

Abstract

We suggest a new technique to construct Markov processes by means of products of copula functions, in the spirit of Darsow et al,(1992). The approach requires to define: i) a sequence of distribution functions of the increments of the process; ii)a sequence of copula functions representing dependence between each increment of the process and the corresponding level of the process before the increment. We show how to use the approach to build symmetric processes, martingale processes, and how to extend the analysis to the multivariate setting. The technique turns out to be well suited to provide a discrete time representation of the dynamics of innovations to financial prices under the restrictions imposed by the Efficient Market Hypothesis.

Keywords: Markov processes, Copula function, Efficient Market Hypothesis, Granger causality, H-condition.

1 Introduction

Finance and physics are the fields in which the theory of stochastic processes have seen the largest development. In finance, in particular, the model of price dynamics first proposed by Bachelier (1900) has become well known as the *Efficient Market Hypothesis* (EMH). In a nutshell, a market is called efficient if price changes are not predictable. In its modern version (see Fama, 1975 and

*University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094370; Fax: +(39) 0512094357. e-mail: umberto.cherubini@unibo.it

†University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094368; Fax: +(39) 0512094357. e-mail: sabrina.mulinacci@unibo.it

‡University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094340; Fax: +(39) 0512094357. e-mail: silvia.romagnoli@unibo.it

Samuelson 1963,1973a, 1973b), the model is applied to the logarithm of prices instead of prices themselves as in the original Bachelier work. In other words, in order to constrain a price S_i, (where i denotes time) to be non-negative, we model the process X_i, which is linked to the price by the relationship

$$S_i = e^{X_i}$$

So, under EMH the innovations of the variable X_i cannot be forecasted on the basis of current and past values of the variable itself: this is called *weak form efficiency*. If innovations cannot be predicted on the basis of other public information either, the market is said to exhibit *semi-strong efficiency*. If private information is also useless the market is said to be *strongly efficient*. So, according to the EMH the price dynamics can be represented by the model

$$X_i = X_{i-1} + Y_i$$

where Y_1 represents the innovation, that is the increment of the log-price, at time i. *Weak form efficiency* results from two requirements: i) X is endowed with the Markov property, that is $\Pr(X_i \mid X_{i-1}, ..., X_0) = \Pr(X_i \mid X_{i-1})$ and ii) the conditional expectation of the increments is equal to zero, that is $E(Y_i \mid X_{i-1}) = 0, \forall i$, that is called the *martingale property*. In more general forms of efficiency the martingale condition is required to hold with respect to larger filtrations: this is called *H-condition*.

In the literature, the EMH is enforced by assuming independent increments, namely with Lévy and additive processes, which allows a synthetic representation in continuous time. In this paper we propose a representation in discrete time that could exploit the flexibility of copulas. As for the Markov property, a technique to represent Markov processes in terms of copulas was first proposed by Darsow et al. (1992). They show that Markov processes can be written as

$$C_{X_{j_1}, X_{j_2}, ..., X_{j_n}} = C_{X_{j_1}, X_{j_2}} \star C_{X_{j_2}, X_{j_3}} \star ... \star C_{X_{j_{n-1}}, X_{j_n}}$$

where the $\star$-product operator is defined as

$$A \star B(u, w, v) \equiv \int_0^w \frac{\partial A(u,t)}{\partial t} \frac{\partial B(t,v)}{\partial t} \, dt \tag{1}$$

for arbitrary bivariate copula functions A and B. This operator is nothing but a way to write the Chapman-Kolmogorov equation in the language of copulas. Ibragimov (2005,2007) extended the representation to the case of Markov processes of order k. The same results can obviously be applied to represent a Markov process in k dimensions, and in that sense will be used in this paper.

While much has already been done to provide a copula-based representation of Markov processes, no result is available, to the best of our knowledge, concerning the possibility to discriminate Markov processes with independent and dependent increments. Among the examples provided in the original Darsow et. al (1992) paper the Brownian copula

$$C(u, v) = \int_0^u \Phi \left(\frac{\sqrt{t}\Phi^{-1}(u) - \sqrt{s}\Phi^{-1}(v)}{\sqrt{t-s}} \right)$$

is well known to represent a dynamics with independent increments, and it is the model applied by Cherubini and Romagnoli (2009) to the evaluation of multivariate barrier derivatives. The interesting question addressed in this paper is how this representation can be generalized to represent all Markov processes with independent increments and how the construction can be amended to extend the class to Markov processes with dependent increments. Apart from this theoretical innovation, we are then interested in investigating whether it is possible to design stochastic processes with dependent increments that abide by the requirements of the EMH, and how the construction can be extended to the multivariate case.
The plan of the paper is as follows. In section 2 we present the technique used to construct Markov processes with independent and dependent increments. In section 3 we show how to apply the technique to build symmetric processes and martingale processes. Section 4 provides the multivariate extension of the analysis. Section 5 concludes.

2 Copula-Based Markov Processes with (In)Dependent Increments

We assume a probability space $\{\mathbf{\Omega}, \Im, \mathbb{P}\}$ and a sequence of random variables $\{Y_n\}_{n\geq 1}$. We define a discrete time stochastic process $\{X_n\}_{n\geq 0}$ through $X_i = X_{i-1} + Y_i$, assuming, by sake of simplicity, $X_0 = 0$. Moreover, we endow the probability space with a filtration $\{\Im_n\}_{n\geq 0}$ (with $\Im_0$ trivial) to which $\{X_n\}_{n\geq 0}$ is adapted. We denote F_{Y_i} the cumulated distribution function of the increment Y_i and F_{X_i} the cumulated distribution function of X_i. Of course, we have $F_{Y_1} = F_{X_1}$. We also assume a set of copula functions $C_i = C_{X_{i-1},Y_i}$ representing the dependence structure between the value of the process at the beginning of the period $[i-1, 1]$ and its increment in that period. Our task is to determine the temporal dependence structure between X_{i-1} and X_i. The representation of bivariate distributions will be then endowed with all the flexibility granted by copula functions

$$\Pr(X_{i-1} \leq x, X_i \leq y) = C_{X_{i-1},X_i}(F_{X_{i-1}}(x), F_{X_i}(y)). \tag{2}$$

Finally, since the stochastic process $\{X_n\}_{n\geq 0}$ is a first order Markov processes, we may apply the result of Darsow et al. (1992) to give a complete description of the law of the process. Before doing that, we start showing how to use copula to model the dependence structure of increments.

2.1 Modelling the Dependence Structure of Increments

Let X, Y be two random variables with continuous c.d.f. F_X and F_Y, respectively, and let $C_{X,Y}(w, \lambda)$ be the copula function that describes their mutual dependence. Even though the approach is general and may be applied to every couple of variables, in what follows Y will denote the increment of the process and X its level before the increment.

We begin by reminding a standard result of the copula functions literature, stating that the partial derivative of a copula function corresponds to the conditional probability distribution. We will adopt the notation $D_iC(u,v)$, $i=1,2$ to represent the partial derivative of a copula function with respect to the i-th argument. Formally, we have that, for every $x,y \in \mathbb{R}$,

$$D_1C_{X,Y}\left(F_X(X),F_Y(y)\right) \text{ is a version of } \mathbb{P}\left(Y \leq y|X\right) \text{ and}$$

$$D_2C_{X,Y}\left(F_X(x),F_Y(Y)\right) \text{ is a version of } \mathbb{P}\left(X \leq x|Y\right).$$

In what follows we will also need the definition of generalized inverse, that for every continuous c.d.f. F we remind to be

$$F^{-1}(w) = \inf\left\{l \in \mathbb{R} : F(l) \geq w\right\}. \tag{3}$$

We are now ready to prove the central proposition that will be used to represent Markov processes with dependent and independent increments. Before doing that, we need to establish a technical result.

Lemma 2.1. *Let $C(u,v)$ be a copula function and F_X and F_Y be two continuous c.d.f., then $\forall t$ a.e. $D_1C\left(\omega, F_Y(t-F_X^{-1}(\omega))\right)$ is well defined $\forall\omega$ a.s.*

Proof. Let us assume that the thesis is false and define

$$N = \{(\omega,v) : D_1C(\omega,v) \text{ is not defined}\}.$$

It follows that an interval (t_0,t_1) with $t_0 < t_1$ exists, such that $\forall t \in (t_0,t_1)$ a.e. it exits $A_t = (a_t,b_t) \subset [0,1], a_t < b_t$ exists, for which, $\forall\omega \in A_t$ a.e. $(\omega, F_Y(t-F_X^{-1}(\omega))) \in N$. Then $(a,b) \subset [0,1]$ exists for which an interval $(\hat{t},\hat{t}+h) \subset (t_0,t_1)$ is defined, such that $(a,b) \subset A_t, \forall t \in (\hat{t},\hat{t}+h)$ a.e.
If $\hat{\omega} \in (a,b)$ exists such that $\exists[s,s+\epsilon] \subset (\hat{t},\hat{t}+h)$ where $F_Y(t-F_X^{-1}(\hat{\omega}))$ is strictly increasing with respect to t, this would contradict the existence a.e. of the function $v \to D_1C(u,v)$ (see [12], Theorem 2.2.7).
On the other hand, if $\forall\omega \in (a,b)$, $F_Y(t-F_X^{-1}(\omega))$ is constant with respect to $t \in (\hat{t},\hat{t}+h)$, putting $v_t(\omega) = F_Y(t-F_X^{-1}(\omega))$, we get

$$\begin{aligned}
&F_Y(\hat{t}-F_X^{-1}(\omega)) = F_Y(\hat{t}+h-F_X^{-1}(\omega)) \quad \forall\omega \in (a,b)\\
&\Rightarrow \mathbb{P}(\hat{t}-F_X^{-1}(\omega) < Y \leq \hat{t}+h-F_X^{-1}(\omega)) = 0 \quad \forall\omega \in (a,b)\\
&\Rightarrow \mathbb{P}(\hat{t}-F_X^{-1}(b) < Y \leq \hat{t}+h-F_X^{-1}(a)) = 0\\
&\Rightarrow \bar{v} = F_Y(\hat{t}-F_X^{-1}(b)) = F_Y(\hat{t}+h-F_X^{-1}(a)) =\\
&= F_Y(t-F_X^{-1}(\omega)) \quad \forall t \in (\hat{t},\hat{t}+h), \forall\omega \in (a,b)\\
&\Rightarrow \forall t \in (\hat{t},\hat{t}+h), v_t(\omega) = \bar{v}
\end{aligned}$$

But in this case $D_1C(u,v)$ will not exists in all the points $(u,\bar{v})$ with $u \in (a,b)$ and this will contradict the existence for all u a.e. of $D_1C(u,v)$ $\forall v$ (see [12],Theorem 2.2.7). □

Proposition 2.1. *Let X e Y be two real-valued random variables on the same probability space $(\Omega, \Im, \mathbb{P})$ with corresponding copula $C_{X,Y}$ and continuous marginals F_X and F_Y. Then,*

$$C_{X,X+Y}(u,v) = \int_0^u D_1 C_{X,Y}\left(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))\right) dw \tag{4}$$

and

$$F_{X+Y}(t) = \int_0^1 D_1 C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw. \tag{5}$$

Proof.

$$\begin{aligned} F_{X,X+Y}(s,t) &= \mathbb{P}\left(X \leq s, X+Y \leq t\right) = \\ &= \int_{-\infty}^{s} \mathbb{P}\left(X+Y \leq t | X = x\right) dF_X(x) = \\ &= \int_{-\infty}^{s} \mathbb{P}\left(Y \leq t - x | X = x\right) dF_X(x) = \\ &= \int_{-\infty}^{s} D_1 C_{X,Y}\left(F_X(x), F_Y(t-x)\right) dF_X(x) = \\ &= \int_0^{F_X(s)} D_1 C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw \end{aligned}$$

where we made the substitution $w = F_X(x) \in (0,1)$.

Then, the copula function linking X and $X+Y$ is

$$\begin{aligned} C_{X,X+Y}(u,v) &= F_{X,X+Y}\left(F_X^{-1}(u), F_{X+Y}^{-1}(v)\right) = \\ &= \int_0^u D_1 C_{X,Y}\left(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))\right) dw. \end{aligned}$$

Moreover

$$F_{X+Y}(t) = \lim_{s \to +\infty} F_{X,X+Y}(s,t) = \int_0^1 D_1 C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw.$$

□

The copula function that is obtained from the proposition is explicitly constructed from the conditional distribution of increments. We can however provide a general proof that equations (4) and (5) jointly describe a copula function. For this purpose, we formally provide an extended definition of the convolution operator.

Definition 2.1. *Let F, H be two continuous c.d.f. and C a copula function. We define the* **C-convolution** *of H and F the c.d.f.*

$$H \overset{C}{*} F(t) = \int_0^1 D_1 C\left(w, F(t - H^{-1}(w))\right) dw$$

Proposition 2.2. *Let F, G, H be three continuous c.d.f., $C(w,\lambda)$ a copula function and*

$$\hat{C}(u,v) = \int_0^u D_1C\left(w, F(G^{-1}(v) - H^{-1}(w))\right) dw.$$

$\hat{C}(u,v)$ is a copula function iff

$$G = H \overset{C}{*} F. \tag{6}$$

Proof. Let us assume (6) to hold. Since there exists a probability space and two random variables X and Y with joint distribution function $F(x,y) = C(F_X(x), F_Y(y))$, thanks to Proposition 2.1, $\hat{C}$ is a copula function.

Vice versa, let $\hat{C}$ be a copula function. Necessarily $\hat{C}(1,v) = v$ holds. But

$$\begin{aligned}\hat{C}(1,v) &= \int_0^1 D_1C\left(w, F\left(G^{-1}(v) - H^{-1}(w)\right)\right) dw = \\ &= H \overset{C}{*} F\left(G^{-1}(v)\right)\end{aligned}$$

and

$$H \overset{C}{*} F\left(G^{-1}(v)\right) = v$$

for all $v \in (0,1)$ if and only if $G = H \overset{C}{*} F$. □

We may then formally define the class of copula functions that we use to construct Markov processes as follows.

Definition 2.2. *Let F and H be two continuous c.d.f. and C a copula function. We define the copula function*

$$\hat{C}(u,v) = \int_0^u D_1C\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw. \tag{7}$$

Remark 2.1. *The C-convolution operator is closed with respect to mixtures of copula functions. In fact, it is trivial to show that for all bivariate copula functions A and B, if $C(u,v) = \lambda A(u,v) + (1-\lambda)B(u,v)$ for $\lambda \in [0,1]$, then, for all c.d.f H and F,*

$$H \overset{C}{*} F = H \overset{\lambda A + (1-\lambda)B}{*} F = \lambda H \overset{A}{*} F + (1-\lambda) H \overset{B}{*} F. \tag{8}$$

It is likewise trivial to convince oneself that this is not also true for the corresponding copula function $\hat{C}(u,v)$ defined through (7). Anyway, we have

$$\begin{aligned}\hat{C}(u,v) = \lambda &\int_0^u D_1A\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw + \\ &+ (1-\lambda)\int_0^u D_1B\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw\end{aligned} \tag{9}$$

with $H \overset{C}{*} F$ given by (8).

This approach produces a natural distinction of stochastic processes depending on whether they evolve by dependent or independent increments. Examples are given below.

2.2 Building Markov processes by increments aggregation

The analysis in the previous sections allows to characterize the law of a stochastic process specifying the distributions of increments and the copula functions expressing the dependence structure between the process at any time i and its increment.

More precisely, if H is the c.d.f. of X_{i-1}, F the c.d.f. of $X_i - X_{i-1}$ and $C(u,v)$ the copula associated to the random vector $(X_{i-1}, X_i - X_{i-1})$, then $H \overset{C}{*} F$ is the c.d.f. of X_i and $\hat{C}(u,v)$, given by (7), is the copula function associated to the random vector (X_{i-1}, X_i).

Building on this, we introduce an iterative technique to construct discrete time Markov process given the distributions of increments and the copula linking the stochastic process at any time to its increment. Following the notation introduced in Section 2, as a consequence of the previous results the temporal dependence structure between X_{i-1} and X_i is given by (see (4))

$$C_{X_{i-1},X_i}(u,v) = \int_0^u D_1 C_i\left(w, F_{Y_i}(F_{X_i}^{-1}(v) - F_{X_{i-1}}^{-1}(w))\right) dw,$$

where by (5)

$$F_{X_i}(x) = F_{X_{i-1}} \overset{C}{*} F_{Y_i}(x) = \int_0^1 D_1 C_i\left(w, F_{Y_i}(x - F_{X_{i-1}}^{-1}(w))\right) dw$$

with, as above, $C_i = C_{X_{i-1},Y_i}$. Finally, if we assume that the process is first order Markov, its dynamics can then be completely described by the sequence of distribution F_{X_i} described above and the sequence of copulas C_{X_{i-1},X_i}.

Example 2.1. The co-monotonic case

In the case $C(w,\lambda) = w \wedge \lambda$, it is easy to verify

$$\begin{aligned} H \overset{C}{*} F(t) &= \int_0^1 \mathbb{I}_{(0,F(t-H^{-1}(w)))}(w) dw = \\ &= \int_0^1 \mathbb{I}_{\{w: F^{-1}(w)+H^{-1}(w)<t\}}(w) dw = \\ &= \sup\left\{w \in (0,1) : F^{-1}(w) + H^{-1}(w) < t\right\} \end{aligned}$$

that implies the well known result

$$F^{-1}(H \overset{C}{*} F(t)) + H^{-1}(H \overset{C}{*} F(t)) = t.$$

Moreover

$$
\begin{aligned}
\hat{C}(u,v) &= \int_0^u \mathbb{I}_{(0,F((H\overset{C}{*}F)^{-1}(v)-H^{-1}(w)))}(w)dw = \\
&= \int_0^u \mathbb{I}_{\{w:H^{-1}(w)+F^{-1}(w)<(H\overset{C}{*}F)^{-1}(v)\}}(w)dw = \\
&= u \wedge \sup\left\{w \in (0,1) : F^{-1}(w) + H^{-1}(w) < (H \overset{C}{*} F)^{-1}(v)\right\} = \\
&= u \wedge v.
\end{aligned}
$$

Example 2.2. The independence case.

If C is the product copula, the C-convolution of H and F coincides with the convolution $H * F$ of H and F, while the copula $\hat{C}$ defined through (7) takes the form

$$
\hat{C}(u,v) = \int_0^u F((H{*}F)^{-1}(v) - H^{-1}(w))dw. \tag{10}
$$

In this case, through our construction, we recover the law of all random walks.

3 Building specific processes

3.1 Symmetric Processes

A natural question is how to build processes endowed with particular features, such as symmetry. The issue of building symmetric processes in the Darsow et al. (1992) framework was addressed in Cherubini and Romagnoli (2009). Here we provide the corresponding characterization for our construction based on increments. The result is quite straightforward.

Proposition 3.1. *If $\overline{C}(u,v) = C(u,v)$ and $\overline{F}_X(t) = F_X(-t), \overline{F}_Y(t) = F_Y(-t)$, then $\overline{F}_{X+Y}(t) = F_{X+Y}(-t)$.*

Proof.

$$
\begin{aligned}
\overline{F}_{X+Y}(t) &= \int_0^1 D_1\overline{C}\left(w, \overline{F}_Y(t - \overline{F}_X^{-1}(w))\right)dw \\
&= \int_0^1 D_1 C\left(w, F_Y(-t + \overline{F}_X^{-1}(w))\right)dw \\
&= \int_0^1 D_1 C\left(w, F_Y(-t - F_X^{-1}(w))\right)dw = F_{X+Y}(-t)
\end{aligned}
$$

since $\overline{F}_X^{-1}(w) = F_X^{-1}(1-w) = -F_X^{-1}(w)$. □

3.2 The Martingale Condition

In this section, we want to impose the martingale restriction to Markov processes.To the best of our knowledge, this topic was first introduced in the Darsow et al. (1992) framework by Ibragimov (2005). Here we introduce the same requirement in our setting, based on modelling increments. Formally, we want to choose the stochastic process for $\{X_i\}_{i\geq 0}$ such that

$$E\left(f(X_{i-1})(X_i - X_{i-1})\right) = 0 \tag{11}$$

for $i \geq 1$ and all Borel measurable functions f.

We want to work out the restrictions that ought to be imposed to copula based representations of Markov processes in order to ensure that condition (11) holds. Actually, our strategy to model increments makes the analysis tractable. For some class of processes, it is definitely immediate. As for processes with independent increments (see Example 2.2), the result follows directly from the requirement.

Proposition 3.2. *Any process whose increments $Y_i \equiv X_i - X_{i-1}$, are independent of the corresponding initial values ($C_{X_{i-1},Y_i}(u,v) \equiv uv$) and their distributions F_{Y_i} have zero mean is a martingale.*

Furthermore, our choice to model the dependence structure between increments and levels provides a straightforward extension to the more general case, in which the independence assumption is dropped. Actually, our entire strategy for the construction of Markov processes is built upon the idea of modeling

$$\mathbb{P}\left(X_i - X_{i-1} \leq x | X_{i-1}\right). \tag{12}$$

It is for this reason that it suffices to concentrate on the copula function $C_{X_{i-1},Y_i}(u,v)$.

Theorem 3.1. *Let $X = (X_i)_{i\geq 0}$ be a Markov process and set $Y_i = X_i - X_{i-1}$. X is a martingale if and only if:*

1. *F_{Y_i} has finite mean for every i;*
2. *for $i \geq 1$, $\int_0^1 F_{Y_i}^{-1}(v) d(D_1 C_{X_{i-1},Y_i}(u,v)) = 0, \quad \forall u \in [0,1]$ a.e..*

Proof. X is a Markov process, to which we impose the condition $\mathbb{E}\left[X_i - X_{i-1}|X_{i-1}\right] = 0$ for every $i \geq 1$. But

$$\begin{aligned}\mathbb{E}\left[X_i - X_{i-1}|X_{i-1}\right] &= \int_{-\infty}^{+\infty} z d\mathbb{P}\left(X_i - X_{i-1} \leq z|X_{i-1}\right) = \\ &= \int_{-\infty}^{+\infty} z d\left(D_1 C_{X_{i-1},Y_i}\left(F_{X_{i-1}}(X_{t_{i-1}}), F_{Y_i}(z)\right)\right) = \\ &= \int_0^1 F_{Y_i}^{-1}(v) d\left(D_1 C_{X_{i-1},Y_i}\left(F_{X_{i-1}}(X_{i-1}), v\right)\right).\end{aligned}$$

The thesis follows setting $F_{X_{i-1}}(X_{i-1}) = u$. □

3.3 Martingale with Symmetric Increments

The above theorem provides the most general set of requirements that have to be imposed to the Markov process to make it a martingale. The interesting question is whether this definition accommodates other classes of processes beyond the independent increment class. In order to construct other cases we first define a class of copula functions.

Definition 3.2. *A copula function $C(u,v)$ is said to be "symmetric around the first coordinate" (or directly symmetric, in this paper), if*

$$\hat{C}(u,v) \equiv u - C(u,1-v) = C(u,v) \tag{13}$$

This concept of symmetry, coupled with symmetry of the distribution of increments, enables us to define an interesting class of martingale processes.

Proposition 3.3. *The martingale condition is satisfied for every symmetric distribution of increments F_{Y_i} if and only if the copula between the increments and the levels is symmetric (around the first coordinate).*

Proof. See in the Appendix. □

A question remains as to how large is the class of symmetric copulas that could be applied in Proposition 3.3. Actually this class may be quite large, since, as we prove below, a copula with the required symmetry feature can be built starting from any arbitrary copula. The same result is found in an even more general setting in Klement, Mester and Pap (2002), who show that this technique can be further extended to all concepts of symmetry, including radial symmetry.

Proposition 3.4. *Take any bivariate copula $A(u,v)$ and its symmetric part $\hat{A}(u,v) \equiv u - A(u,1-v)$. Define: $C(u,v) \equiv 0.5A(u,v) + 0.5\hat{A}(u,v)$. Then, $C(u,v)$ is a copula and it is symmetric in the sense that $C(u,v) = \hat{C}(u,v)$.*

Proof. First, notice that it is easy to show that $\hat{A}(u,v)$ is a copula (see Nelsen, 2006). Second, $C(u,v)$ is a copula because it is a mixture of copulas. It may be in fact immediately verified that $C(0,v) = C(u,0) = 0$, $C(1,v) = v$, $C(u,1) = u$. It is 2-increasing because it is the sum of two increasing elements. Having proved that it is a copula, the symmetry property of $C(u,v)$ can be easily checked

$$\begin{aligned} \hat{C}(u,v) &= u - C(u,1-v) \\ &= u - (0.5A(u,1-v) + 0.5u - 0.5A(u,v)) \\ &= 0.5A(u,v) + 0.5u - 0.5A(u,1-v) = C(u,v) \end{aligned} \tag{14}$$

□

Proposition 3.4 states that all symmetric copulas (in our sense) can be obtained in this way. For every choice of the class of symmetric distributions of increments we can then choose a symmetric copula function $C(u,v)$ corresponding to an arbitrary copula function $A(u,v)$. Furthermore, all the copulas endowed with this symmetry property can be represented by this procedure.

4 Copula characterization of bivariate Markov processes

We now extend the above analysis to the case of multivariate setting. We firstly provide extension of the copula approach to Markov processes to a multivariate setting following Ibragimov (2007).

Let $m, n \geq 2$ and A and B be, respectively, m- and n- dimensional copulas. Set

$$A_{1,\ldots,m|m-1,m}(u_1,\ldots,u_{m-2},\xi,\eta) = \frac{\frac{\partial^2 A(u_1,\ldots,u_{m-2},\xi,\eta)}{\partial\xi\partial\eta}}{\frac{\partial^2 A(1,\ldots,1,\xi,\eta)}{\partial\xi\partial\eta}}$$

and

$$B_{1,\ldots,n|1,2}(\xi,\eta,u_3,\ldots,u_n) = \frac{\frac{\partial^2 B(\xi,\eta,u_3,\ldots,u_n)}{\partial\xi\partial\eta}}{\frac{\partial^2 B(\xi,\eta,1\ldots,1)}{\partial\xi\partial\eta}}.$$

If

$$A(1,\ldots,1,\xi,\eta) = B(\xi,\eta,1,\ldots,1) = C(\xi,\eta)$$

where C is a bivariate copula, we can define the $\star^2$-product of the copulas A and B as the copula $D = A \star^2 B : [0,1]^{m+n-2} \to [0,1]$ given by

$$\begin{aligned} D(u_1,\ldots,u_{m+n-2}) = \int_0^{u_{m-1}} \int_0^{u_m} & A_{1,\ldots,m|m-1,m}(u_1,\ldots,u_{m-2},\xi,\eta) \cdot \\ & \cdot B_{1,\ldots,n|1,2}(\xi,\eta,u_3,\ldots,u_n) dC(\xi,\eta). \end{aligned} \tag{15}$$

The $\star^2$ operator is a particular case of the $\star^k$ operator defined in Ibragimov (2007).

Recall that, if $(Y_1,\ldots,Y_n)$ is a random vector with associated copula function $C(u_1,\ldots,u_n)$ and margins F_i for $i = 1,\ldots,n$,

$$\mathbb{P}(Y_1 \leq y_1,\ldots,Y_{n-2} \leq y_{n-2} | Y_{n-1} = x, Y_n = y) = \frac{\frac{\partial^2 C(F_1(y_1),\ldots,F_{n-2}(y_{n-2}),F_{n-1}(x),F_n(y))}{\partial u_{n-1}\partial u_n}}{\frac{\partial^2 C(1,1,\ldots,1,F_{n-1}(x),F_n(y))}{\partial u_{n-1}\partial u_n}}$$

that is

$$\mathbb{P}(Y_1 \leq y_1,\ldots,Y_{n-2} \leq y_{n-2} | Y_{n-1} = x, Y_n = y) = C_{1,\ldots,n|n-1,n}(F_1(y_1),\ldots,F_{n-1}(x),F_n(y))$$

and, similarly,

$$\mathbb{P}(Y_3 \leq y_3,\ldots,Y_n \leq y_n | Y_1 = x, Y_2 = y) = C_{1,\ldots,n|1,2}(F_1(x),F_2(y),\ldots,F_{n-1}(y_{n-1}),F_n(y_n)).$$

Let $(X, Z) = \{(X_i, Z_i)\}_{i\geq 0}$ be an $\mathbb{R}^2$-valued stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_i^{X,Z})_{i\geq 0}$ be its natural filtration.

By definition (X, Z) is a Markov process if, for all i, such that $j_1 < \ldots < j_i < \ldots < j_{n+1}$ and all $(x_{n+1}, z_{n+1}) \in \mathbb{R}^2$

$$\begin{aligned}&\mathbb{P}\left(X_{n+1} \leq x_{n+1}, Z_{n+1} \leq z_{n+1} | X_n, Z_n, X_{n-1}, Z_{n-1}, \ldots, X_1, Z_1\right) = \\ &= \mathbb{P}\left(X_{n+1} \leq x_{n+1}, Z_{n+1} \leq z_{n+1} | X_n, Z_n\right).\end{aligned} \tag{16}$$

Let $C_{j_1,j_2,\ldots,j_n}(u_1, v_1, \ldots, u_n, v_n)$ denote the $2n$-dimensional copulas corres ponding to the joint distribution of the random vector $(X_1, Z_1, X_2, Z_2, \ldots, X_n, Z_n)$.

We set

$$C_{j_1,\ldots,j_n|j_n}(u_1, v_1, \ldots, u_n, v_n) = \frac{\frac{\partial^2 C(u_1,v_1,\ldots,u_{n-1},v_{n-1},\xi,\eta)}{\partial\xi\partial\eta}}{\frac{\partial^2 C(1,\ldots,1,\xi,\eta)}{\partial\xi\partial\eta}}$$

and

$$C_{j_1,\ldots,j_n|j_1}(\xi, \eta, \ldots, u_n, v_n) = \frac{\frac{\partial^2 C(\xi,\eta,\ldots,u_n,v_n)}{\partial\xi\partial\eta}}{\frac{\partial^2 C(1,\ldots,1,\xi,\eta)}{\partial\xi\partial\eta}}.$$

Theorem 4.1. *An $\mathbb{R}^2$-valued stochastic process (X, Z) is a Markov process if and only if for all i, such that $j_1 < \ldots < j_i < \ldots < j_n$*

$$C_{j_1,j_2,\ldots,j_n} = C_{j_1,j_2} \star^2 C_{j_2,j_3} \star^2 \cdots \star^2 C_{j_{n-1},j_n}. \tag{17}$$

Proof. See in the Appendix. □

4.1 The Martingale Condition

An important feature of innovation modelling, when applied to the asset price dynamics, is the martingale condition, that makes such innovations unpredictable. To the best of our knowledge, this topic was first introduced in the Darsow et al. (1992) framework by Ibragimov (2005). Here we introduce the same requirement in our setting, based on modelling increments.
By definition, (X, Z) is a martingale with respect to $\mathcal{F}^{X,Z}$ iff

$$\mathbb{E}\left[X_{i+1} - X_i | X_i, Z_i\right] = 0, \quad \forall i \geq 0$$

and

$$\mathbb{E}\left[Z_{i+1} - Z_i | X_i, Z_i\right] = 0, \quad \forall i \geq 0.$$

Let $\Delta X_i = X_{i+1} - X_i$ and $\Delta Z_i = Z_{i+1} - Z_i$ and $A_{i,i+1}(u, v, w, \lambda)$ be the copula function of the random vector $(X_i, Z_i, \Delta X_i, \Delta Z_i)$ and $F_{X_i}, F_{Z_i}, F_{\Delta X_i}, F_{\Delta Z_i}$ the corresponding marginal c.d.fs.. We set $D_{12}A_{i,i+1}(u, v, w, \lambda) = \frac{\partial^2}{\partial u \partial v} A_{i,i+1}(u, v, w, \lambda)$.

Theorem 4.2. *The Markov process (X,Z) is a martingale with respect to the filtration $\mathcal{F}^{X,Z}$ iff:*

1. *$F_{\Delta X_i}$ and $F_{\Delta Z_i}$ have finite mean for every i;*
2. *for every i,*

$$\int_0^1 F^{-1}_{\Delta X_i}(w) d\left(D_{1,2}A_{i,i+1}(u,v,w,1)\right) = 0, \quad \forall u,v \in [0,1] \tag{18}$$

and

$$\int_0^1 F^{-1}_{\Delta Z_i}(w) d\left(D_{1,2}A_{i,i+1}(u,v,1,w)\right) = 0, \quad \forall u,v \in [0,1]. \tag{19}$$

Proof. See in the Appendix. □

4.2 No-Granger causality

We now show that once the martingale condition has been proved for each process, the multivariate extension can be recovered by simply applying a concept that is standard in econometrics and is known as Granger causality.

Definition 4.3. *Let $X = (X_i)$ and $Z = (Z_i)$ be two stochastic processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_i^X$ and $\mathcal{F}_i^Z$ be their respective natural filtrations, while $\mathcal{F}_i^{X,Z}$ denotes the natural filtration of the bivariate process $(X,Z) = (X_i, Z_i)$.*

We say that Z_i doesn't cause X_i with respect to $\mathcal{F}_i^{X,Z}$ if

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^{X,Z}] = \mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^X]$$

for any i and x.

Remark 4.1. *Definition 4.3 is the usual concept of noncausality in the sense of Granger and for this reason we call this "no-Granger causality".*

Remark 4.2. *We recall that the concept of noncausality allows to analyze the stability of a martingale property with respect to increasing filtrations. More specifically it is proved (see Brémaud and Yor, 1978, Florens and Fugère, 1991)that if X and Z are two stochastic processes and X is an $\mathcal{F}_i^X$- martingale , it is an $\mathcal{F}_i^{X,Z}$-martingale as well iff Z does not Granger cause X for every i.*

Let us now restrict the analysis to the class of Markov processes. Remember that a process, Markov with respect to a given filtration, is not in general Markov with respect to a larger filtration. We show that this is in fact guaranteed by no-Granger causality.

Theorem 4.4. *The following are equivalent:*

1. Z does not Granger cause X for every i;

2. if X is an $\mathcal{F}_i^X$-Markov process, then it is an $\mathcal{F}_i^{X,Z}$-Markov process, as well.

Proof. 1. $\Rightarrow$ 2.:1. implies

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^{X,Z}] = \mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^X].$$

for every $x \in \mathbb{R}$. By hypothesis

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x | X_i]$$

and the thesis follows. The other implication is trivial. □

We saw that no-Granger casuality and Markov property of each process with respect to its natural filtration implies the Markov structure of the system. The converse does not hold as the following Remark shows.

Remark 4.3. *In fact, let (X, Z) be a Markov process with respect to its natural filtration so that*

$$\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X,Z}\right) = \mathbb{P}\left(X_{i+1} \leq x, | X_i, Z_i\right) \tag{20}$$

If Z does not Granger cause X for every i,

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^{X,Z}] = \mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x | X_i, X_{i-1}, X_{i-2}, \ldots, X_0] \tag{21}$$

for every $x \in \mathbb{R}$.

(20) and (21) do not imply that X is a Markov process with respect to its natural filtration. Take $X_{i+1} = Z_i + X_i$ and $Z_{i+1} = X_i$ as a counterexample: (X, Z) is a Markov process, X satisfies (20) and (21), but

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x | X_i, X_{i-1}, X_{i-2}, \ldots, X_0] \neq \mathbb{P}[X_{i+1} \leq x | X_i].$$

In order to guarantee that, given a multivariate Markov process, each of its components be a Markov process with respect to its own natural filtration as well, it is necessary to introduce an adequate restriction to the law the processes involved.

Proposition 4.1. *Let (X, Z) be a bivariate Markov process and assume that Z does not Granger cause X for every i.*

If X is an $\mathcal{F}^X$-Markov process, then

$$\mathbb{P}\left(X_{i+1} \leq x | X_i, Z_i\right) = \mathbb{P}\left(X_{i+1} \leq x | X_i\right).$$

Proof. By Hypothesis

$$\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X,Z} \right) = \mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X} \right),$$

$$\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X,Z} \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i, Z_i \right)$$

and

$$\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X} \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i \right).$$

The thesis trivially follows. □

Proposition 4.2. *Let (X,Z) be a bivariate Markov process. If*

$$\mathbb{P}\left(X_{i+1} \leq x | X_i, Z_i \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i \right)$$

then X is an $\mathcal{F}^X$-Markov process.

Proof. By hypothesis

$$\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X,Z} \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i, Z_i \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i \right).$$

But

$$\begin{aligned} \mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X} \right) &= \mathbb{E}\left[\mathbb{P}\left(X_{i+1} \leq x | \mathcal{F}_i^{X,Z} \right) \Big| \mathcal{F}_i^{X} \right] = \\ &= \mathbb{E}\left[\mathbb{P}\left(X_{i+1} \leq x | X_i \right) | \mathcal{F}_i^{X} \right] = \\ &= \mathbb{P}\left(X_{i+1} \leq x | X_i \right). \end{aligned}$$

□

Propositions 4.1 and 4.2 are equivalent to Lemma 3.5 in (Florens et al., 1993).

Theorem 4.5. *Let (X,Z) be a bivariate Markov process and $C_{i,i+1}(u_1, v_1, u_2, v_2) = C_{X_i,Z_i,X_{i+1},Z_{i+1}}(u_1, v_1, u_2, v_2)$.*

$$\mathbb{P}\left(X_{i+1} \leq x | X_i, Z_{t_i} \right) = \mathbb{P}\left(X_{i+1} \leq x | X_i \right)$$

iff

$$C_{i,i+1}(u_1, v_1, u_2, 1) = C_{Z_i,X_i} \star C_{X_i,X_{i+1}}(v_1, u_1, u_2).$$

Proof. See in the Appendix. □

Similarly we obtain that X does not Granger cause Z iff

$$C_{i,i+1}(u_1, v_1, 1, v_2) = C_{X_i,Z_i} \star C_{Z_i,Z_{i+1}}(u_1, v_1, u_2).$$

Remark 4.4. *Notice that if the system can be given a representation as in Theorem 4.5, the copula function* $C_{1,2,\ldots,n}$ *can be in a hierarchical form*

$$C_{1,2,\ldots,n}(u_1, v_1, \ldots, u_n, v_n) = C\left(G(u_1, \ldots, u_n), H(v_1, \ldots, v_n)\right)$$

where the notation means that

$$G(u_1, \ldots, u_n) = C_{1,2} \star C_{2,3} \star \cdots \star C_{n-1,n}(u_1, \ldots, u_n)$$

where $C_{i,i+1}$ *are the copula functions linking* X_i *and* X_{i+1}*. The same representation applies to* H *corresponding to the dependence structure of* Z*.*

5 Conclusions

In this paper we have tackled the problem of constructing Markov processes for speculative prices; in the spirit of the copula-based representation that was first introduced by Darsow et al. (1992). The approach requires to define:

- a sequence of distribution functions of the increments of the process;
- a sequence of copula functions representing dependence between each increment of the process and the corresponding level of the process before the increment.

We find that this construction is very well suited to impose restrictions that are consistent with the speculative price dynamics expected under the Efficient Market Hypothesis. Namely, we specify conditions under which innovations of log prices are unpredictable. More precisely, we single out two classes of Markov processes endowed with this martingale condition:

- processes with independent increments with zero mean distributions;
- processes with symmetric increments linked to the initial levels by a symmetric copula.

We find that the extension of the martingale restriction to a multivariate setting involves a concept which is very well known in econometrics and is called *Granger causality*. We show how to express this concept in our copula based framework.

6 Appendix

Proof of Proposition 3.3. For simplicity we set $C_{X_{i-1},Y_i} = C$ and $F_i = F$. Being F a symmetric distribution,

$$\begin{aligned}
\int_0^1 F^{-1}(v)d(D_1C(u,v)) &= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1C(u,v)) + \int_{\frac{1}{2}}^1 F^{-1}(v)d(D_1C(u,v)) = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1C(u,v)) + \int_{\frac{1}{2}}^0 F^{-1}(1-\rho)d(D_1C(u,1-\rho)) = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1C(u,v)) + \int_0^{\frac{1}{2}} F^{-1}(\rho)d(D_1C(u,1-\rho)) = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1C(u,v)) + d(D_1C(u,1-v)) = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1C(u,v)) + D_1C(u,1-v)) = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)d(D_1(C(u,v) + C(u,1-v)) = 0, \quad \forall u \in (0,1).
\end{aligned}$$

Last condition is satisfied for every symmetric distribution F iff (notice that, in last integral, $F^{-1}(v) < 0$ in an opportune not empty interval)

$$d(D_1(C(u,v) + C(u,1-v))) = 0 \quad \forall u, v \in (0,1).$$

It maybe easily verified that this condition is satisfied if and only if

$$C(u,v) + C(u,1-v) = u$$

which is the symmetry condition assumed for the copula □

Proof of Theorem 4.1. It is trivial to show that property (16) holds if and only if

$$\begin{aligned}
&\mathbb{P}\left(X_1 \le x_1, Z_1 \le y_1, \dots, X_{n-1} \le x_{n-1}, Z_{n-1} \le y_{n-1}, X_{n+1} \le x_{n+1}, Z_{n+1} \le y_{n+1} | \, X_n, Z_n \right) = \\
&= \mathbb{P}\left(X_1 \le x_1, Z_1 \le y_1, \dots, X_{n-1} \le x_{n-1}, Z_{n-1} \le y_{n-1} | \, X_n, Z_n \right) \cdot \\
&\cdot \mathbb{P}\left(X_{n+1} \le x_{n+1}, Z_{n+1} \le y_{n+1} | \, X_n, Z_n \right)
\end{aligned} \tag{22}$$

that is

$$\begin{aligned}
&\mathbb{P}\left(X_1 \le x_1, Z_1 \le y_1, \dots, X_{n-1} \le x_{n-1}, Z_{n-1} \le y_{n-1}, X_{n+1} \le x_{n+1}, Z_{n+1} \le y_{n+1} | \, X_n, Z_n \right) = \\
&= C_{1,\dots,n|t_n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), F_{X_n}(X_n), F_{Z_n}(Z_n) \right) \cdot \\
&\cdot C_{n,n+1|n}\left(F_{X_n}(X_n), F_{Z_n}(Z_n), F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1}) \right).
\end{aligned} \tag{23}$$

Integrating (23) over $X_n^{-1}((-\infty, x_n)) \times Z_n^{-1}((-\infty, y_n))$, we get

$$\begin{aligned}
&C_{1,\dots,n,n+1}(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})) = \\
&= \int_{-\infty}^{x_n}\int_{-\infty}^{y_n} C_{1,\dots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), F_{X_n}(x), F_{Z_n}(y)\right) \cdot \\
&\cdot C_{n,n+1|n}\left(F_{X_n}(x), F_{Z_n}(y), F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dF_{(X_n,Z_n)}(x,y) = \\
&= \int_{-\infty}^{F_{X_n}(x_n)}\int_{-\infty}^{F_{Z_n}(y_n)} C_{1,\dots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), \xi, \eta\right) \cdot \\
&\cdot C_{n,n+1|n}\left(\xi, \eta, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dC_n(\xi,\eta) = \\
&= C_{1,\dots,n} \star^2 C_{n,n+1}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right).
\end{aligned} \tag{24}$$

By induction, we obtain (17).

Conversely, suppose that (17) holds. We have

$$\begin{aligned}
&\mathbb{P}(X_1 \le x_1, Z_1 \le y_1, \dots, X_{n+1} \le x_{n+1}, Z_{n+1} \le y_{n+1}) = \\
&= C_{1,\dots n,n+1}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) = \\
&= \int_{-\infty}^{F_{X_n}(x_n)}\int_{-\infty}^{F_{Z_n}(y_n)} C_{1,\dots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), \xi, \eta\right) \cdot \\
&\cdot C_{n,n+1|n}\left(\xi, \eta, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dC_n(\xi,\eta) = \\
&= \mathbb{E}\left[\mathbb{P}(X_1 \le x_1, Z_1 \le y_1, \dots, X_{n-1} \le x_{n-1}, Z_{n-1} \le y_{n-1} | X_n, Z_n) \cdot\right. \\
&\left.\cdot\, \mathbb{P}(X_{n+1} \le x_{n+1}, Z_{n+1} \le y_{n+1} | X_n, Z_n)\, \mathbb{I}_{\{X_n \le x_n, Z_n \le y_n\}}\right]
\end{aligned}$$

from which (22) follows. □

Proof of Theorem 4.2. Since

$$\mathbb{P}(\Delta X_i \le z | X_i, Z_i) = \frac{D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1)}{D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)},$$

$$\begin{aligned}
&\mathbb{E}[X_{i+1} - X_i | X_i, Z_i] = \int_{-\infty}^{+\infty} z d\mathbb{P}(\Delta X_i \le z | X_i, Z_i) = \\
&= \int_{-\infty}^{+\infty} z d\left(\frac{D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i(z),1})}{D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)}\right) = \\
&= \frac{1}{D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)} \int_{-\infty}^{+\infty} z d\left(D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1)\right) = 0
\end{aligned}$$

if and only if

$$\int_{-\infty}^{+\infty} z d\left(D_{1,2}A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1)\right) = 0. \tag{25}$$

Now, setting $F_{X_i}(X_i) = u$, $F_{Z_i}(Z_i) = v$ and $F_{\Delta X_i}(z) = w$, (25) is equivalent to

$$\int_0^1 F^{-1}_{\Delta_h X_t}(w) d\left(D_{1,2} A_{t,t+h}(u,v,w,1)\right) = 0, \qquad \forall u, v \in [0,1].$$

(19) is obtainable similarly. □

Proof of Theorem 4.5. Since

$$\mathbb{P}\left(X_{i+1} \le x | X_i, Z_i\right) = \frac{\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{X_{i+1}}(x), 1\right)}{\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1\right)}$$

and

$$\mathbb{P}\left(X_{i+1} \le x | X_i\right) = \frac{\partial}{\partial u_1} C_{i,i+1}\left(F_{X_i}(X_i), 1, F_{X_{i+1}}(x), 1\right),$$

the no-Granger causality holds iff

$$\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(u_1, v_1, u_2, 1\right) = \frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(u_1, v_1, 1, 1\right) \frac{\partial}{\partial u_1} C_{i,i+1}\left(u_1, 1, u_2, 1\right).$$

Integrating we obtain

$$\begin{aligned} C_{i,i+1}(u_1, v_1, u_2, 1) &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{i,i+1}(u', v_1, 1, 1) \frac{\partial}{\partial u'} C_{i,i+1}(u', 1, u_2, 1) du' = \\ &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{X_i, Z_i}(u', v_1) \frac{\partial}{\partial u'} C_{X_i, X_{i+1}}(u', u_2) du' = \\ &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{Z_i, X_i}(v_1, u') \frac{\partial}{\partial u'} C_{X_i, X_{i+1}}(u', u_2) du' = \\ &= C_{Z_i, X_i} \star C_{X_i, X_{i+1}}(v_1, u_1, u_2). \end{aligned}$$

□

References

[1] Bachelier, L. (1900): *Théorie de la Speculation*, Gauthier-Villard, Paris.

[2] Brémaud, P., Yor, M. (1978): “Changes of Filtrations and of Probability Measures”, *Z. Wahrscheinl*, 45, 269-295

[3] Cherubini, U., S. Romagnoli, (2009), “The Dependence Structure of Running Maxima and Minima: Theoretical Results and Option Pricing Applications”,*Mathematical Finance*, forthcoming

[4] Darsow, Z.F.,B. Nguyen,E.T. Olsen, (1992), ”Copulas and Markov Processes”, *Illinois Journal of Mathematics*, 36, 600-642

[5] Fama E.F. (1965): "Efficient Capital Markets: A Review of Theory and Empirical Work", *Journal of Finance*, **25(2)**, 383-417

[6] Florens, J.P. and D. Fougère, (1991), "Non-causality in continuous time: applications to counting processes", *Cahier du GREMAQ 91.b*, Université des Sciences Sociales de Toulouse

[7] Florens, J.P.,M. Mouchart,J.M. Rolin, (1993), "Noncausality and marginalization of Markov processes", *Econometric Theory*, 9, 241-262

[8] Ibragimov, R., (2005), "Copula based characterization and modeling for time series", *Harvard Institute of Economic Research*, Discussion paper n. 2094

[9] Ibragimov, R., (2007), "Copula-based characterization for higher-order Markov processes", *Econometric Theory*, forthcoming

[10] Klement E.P, R. Mesiar and E. Pap, (2002), "Invariant Copulas", *Kybernetica (Prague)*, 275-285

[11] Mazziotto, G., and Szpirglas (1978), "Modéle général de filtrage non linéaire et équations différentielles stochastiques associées" , *Annals Inst. Henry Poincaré*, **15**,147-173

[12] Nelsen, R. (2006), *Introduction to Copulas,* Springer Verlag

[13] Samuelson P.A. (1963), "Proof That Properly Anticipated Prices Fluctuate Randomly", *Industrial Management Review*, (6), 41-50

[14] Samuelson P.A. (1973a), "Mathematics of Speculative Price", *SIAM Review*, (15(1)), 1-42

[15] Samuelson P.A. (1973b), "Proof That Properly Discounted Present Values of Assets Fluctuate Randomly" *Bell Journal of Economics and Management*,(4(2)), 369-374