

On the distribution of (un)bounded sum of random variables

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Abstract

We propose a general treatment of random variables aggregation accounting for dependence among variables and bounded or unbounded support of their sum. The approach is based on the extension to the concept of convolution to dependent variables, involving copula functions. We show that some classes of copula functions (such as Marshall-Olkin and elliptical) cannot be used to represent the dependence structure of two variables whose sum is bounded, while Archimedean copulas can be applied only if the generator becomes linear beyond some point. As for the application, we study the problem of reinsurance of a portfolio of risks, showing how the insurance policy impact on their dependence structure of losses.

Keywords: Copula functions, sum of dependent random variables, reinsurance.

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1 Introduction

This paper presents a general treatment of random variables aggregation accounting for the following two features: i) dependence between the random variables; ii) bounded or unbounded support of the sum.

It is well known that the distribution of sums of variables is the convolution of marginal distributions in the case of independence, while to the best of our

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knowledge no general characterization has been given to the general case of sums of dependent variables. A partial exception is a stream of literature investigating the bounds of such distribution (see Frank, Nelsen, and Schweizer, 1987, Moynihan, Schweizer and Sklar, 1978, Makarov, 1981 and Denuit, Genest, and Marceau, 1999). Here we propose a method to recover the distribution of sums of variables under general dependence structures. Moreover, we address the issue of how the presence of bounds on the sum can impact on the dependence structure of variables. Our approach is based on the extension to the concept of convolution to dependent variables, involving copula functions.

By accounting for consistency between the support of random variables and that of the sum, we identify the constraints that have to be imposed to the copula function and which allow to determine dependence structures consistent with the aggregation problem.

The plan of the paper is as follows. In section 2 we address the problem of aggregation of random variables in a general setting, considering continuous random variables and assuming that their sum is not subject to any bound. In section 3 we extend the analysis to the case in which the sum is bounded, either from above or from below, we provide examples of dependence structures inconsistent with such bound, as well as examples of (Archimedean) copulas which instead are consistent with it. In section 4 we show how to extend the analysis to the case of discrete random variables. Finally, Section 5 presents an application of the model to insurance, showing how the choice of a reinsurance policy impacts on the dependence structures of the risks in a portfolio. Section 6 concludes.

2 Copula-Based Convolution

In this section we focus on the distribution of the sum of two continuous random variables. Firstly we study the general unbounded setting and then we consider the boundness constraints. Let X, Y be two random variables with continuous c.d.f. F_X and F_Y . Let $C_{X,Y}(w, \lambda)$ be the copula function that describes their mutual dependence.

For every continuous c.d.f. F we define the generalized inverse $F^{-1} : (0, 1) \rightarrow \mathbb{R}$ as

$$F^{-1}(w) = \inf \{l \in \mathbb{R} : F(l) \geq w\}. \quad (1)$$

Lemma 2.1. *Let $C(u, v)$ be a copula function and F_X and F_Y be two continuous c.d.f., then $\forall t$ a.e. $D_1 C(\omega, F_Y(t - F_X^{-1}(\omega)))$ is well defined $\forall \omega$ a.s.*

Proof. Let us assume that the thesis is false and define

$$N = \{(\omega, v) : D_1 C(\omega, v) \text{ is not defined}\}$$

It follows that an interval (t_0, t_1) with $t_0 < t_1$ exists, such that $\forall t \in (t_0, t_1)$ a.e. it exists $A_t = (a_t, b_t) \subset [0, 1], a_t < b_t$ exists, for which, $\forall \omega \in A_t$ a.e.

$(\omega, F_Y(t - F_X^{-1}(\omega))) \in N$. Then $(a, b) \subset [0, 1]$ exists for which an interval $(\hat{t}, \hat{t} + h) \subset (t_0, t_1)$ is defined, such that $(a, b) \subset A_t, \forall t \in (\hat{t}, \hat{t} + h)$ a.e. If $\hat{\omega} \in (a, b)$ exists such that $\exists[s, s + \epsilon] \subset (\hat{t}, \hat{t} + h)$ where $F_Y(t - F_X^{-1}(\hat{\omega}))$ is strictly increasing with respect to t , this would contradict the existence a.e. of the function $v \rightarrow D_1C(u, v)$ (see [?], Theorem 2.2.7). On the other hand, if $\forall \omega \in (a, b)$, $F_Y(t - F_X^{-1}(\omega))$ is constant with respect to $t \in (\hat{t}, \hat{t} + h)$, putting $v_t(\omega) = F_Y(t - F_X^{-1}(\omega))$, we get

$$\begin{aligned}
& F_Y(\hat{t} - F_X^{-1}(\omega)) = F_Y(\hat{t} + h - F_X^{-1}(\omega)) \quad \forall \omega \in (a, b) \\
& \Rightarrow \mathbb{P}(\hat{t} - F_X^{-1}(\omega) < Y \leq \hat{t} + h - F_X^{-1}(\omega)) = 0 \quad \forall \omega \in (a, b) \\
& \Rightarrow \mathbb{P}(\hat{t} - F_X^{-1}(b) < Y \leq \hat{t} + h - F_X^{-1}(a)) = 0 \\
& \Rightarrow \bar{v} = F_Y(\hat{t} - F_X^{-1}(b)) = F_Y(\hat{t} + h - F_X^{-1}(a)) = \\
& = F_Y(t - F_X^{-1}(\omega)) \quad \forall t \in (\hat{t}, \hat{t} + h), \forall \omega \in (a, b) \\
& \Rightarrow \forall t \in (\hat{t}, \hat{t} + h), v_t(\omega) = \bar{v}
\end{aligned}$$

But in this case $D_1C(u, v)$ will not exist in all the points (u, \bar{v}) with $u \in (a, b)$ and this will contradict the existence for all u a.e. of $D_1C(u, v) \forall v$ (see [?], Theorem 2.2.7). \square

Proposition 2.1. *Let X e Y be two real-valued random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with corresponding copula $C(w, \lambda)$ and continuous marginals F_X and F_Y . Then, $\forall t$ a.s.*

$$F_{X+Y}(t) = \int_0^1 D_1C(w, F_Y(t - F_X^{-1}(w))) dw. \quad (2)$$

Proof.

$$\begin{aligned}
F_{X, X+Y}(s, t) &= \mathbb{P}(X \leq s, X + Y \leq t) = \\
&= \int_{-\infty}^s \mathbb{P}(X + Y \leq t | X = x) dF_X(x) = \\
&= \int_{-\infty}^s \mathbb{P}(Y \leq t - x | X = x) dF_X(x) = \\
&= \int_{-\infty}^s D_1C(F_X(x), F_Y(t - x)) dF_X(x) = \\
&= \int_0^{F_X(s)} D_1C(w, F_Y(t - F_X^{-1}(w))) dw
\end{aligned}$$

where we made the substitution $w = F_X(x) \in (0, 1)$ and we used Lemma 2.1. Then

$$F_{X+Y}(t) = \lim_{s \rightarrow +\infty} F_{X, X+Y}(s, t) = \int_0^1 D_1C(w, F_Y(t - F_X^{-1}(w))) dw.$$

\square

For every c.d.f G , let $\bar{G}(x) = 1 - G(x)$ be the survival function and for every copula function C , let $\bar{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ be the survival copula. It is well known that $\bar{F}_{X,Y}(x, y) = \mathbb{P}(X > x, Y > y) = \bar{C}(\bar{F}_X(x), \bar{F}_Y(y))$ and that $\bar{C}(u, v)$ is again a copula (see [?], Section 2.6).

Being \bar{G} nonincreasing, we define the generalized inverse as $\bar{G}^{-1} : (0, 1) \rightarrow \mathbb{R}$ as

$$\bar{G}^{-1}(w) = \inf \{l \in \mathbb{R} : \bar{G}(l) \leq w\}. \quad (3)$$

Clearly, $\bar{G}^{-1}(w) = G^{-1}(1 - w)$.

Corollary 2.1. *Let X e Y be two real-valued random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with corresponding copula $C(w, \lambda)$ and continuous margins F_X and F_Y . Then $\forall t$ a.s.*

$$\bar{F}_{X+Y}(t) = \int_0^1 D_1 \bar{C} \left(w, \bar{F}_Y(t - \bar{F}_X^{-1}(w)) \right) dw. \quad (4)$$

Proof. Notice that, for every copula function C ,

$$\bar{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$$

which is again a copula, so that

$$D_1 \bar{C}(u, v) = 1 - D_1 C(1 - u, 1 - v).$$

Now, thanks to (2), we have

$$\begin{aligned} \bar{F}_{X+Y}(t) &= 1 - \int_0^1 [1 - D_1 \bar{C}(1 - w, \bar{F}_Y(t - \bar{F}_X^{-1}(w)))] dw = \\ &= \int_0^1 D_1 \bar{C}(1 - w, \bar{F}_Y(t - \bar{F}_X^{-1}(1 - w))) dw = \\ &= \int_0^1 D_1 \bar{C}(w, \bar{F}_Y(t - \bar{F}_X^{-1}(w))) dw \end{aligned} \quad (5)$$

□

Corollary 2.2. *If $\bar{C}(u, v) = C(u, v)$ and $\bar{F}_X(t) = F_X(-t), \bar{F}_Y(t) = F_Y(-t)$, then $\bar{F}_{X+Y}(t) = F_{X+Y}(-t)$.*

Proof.

$$\begin{aligned} \bar{F}_{X+Y}(t) &= \int_0^1 D_1 \bar{C} \left(w, \bar{F}_Y(t - \bar{F}_X^{-1}(w)) \right) dw \\ &= \int_0^1 D_1 C \left(w, F_Y(-t + \bar{F}_X^{-1}(w)) \right) dw \\ &= \int_0^1 D_1 C \left(w, F_Y(-t - F_X^{-1}(w)) \right) dw = F_{X+Y}(-t) \end{aligned}$$

since $\bar{F}_X^{-1}(w) = F_X^{-1}(1 - w) = -F_X^{-1}(w)$.

□

Proposition 2.1 allows to give the following definition;

Definition 2.3. Let F, G be two continuous c.d.f. and C a copula function. We define the **C-convolution** of F and G the c.d.f. $F \overset{C}{*} G$ defined by

$$F \overset{C}{*} G(t) = \int_0^1 D_1 C(w, G(t - F^{-1}(w))) dw \quad \forall t \quad a.e.$$

Remark 2.1. Independence

If C is the product copula, the C -convolution of H and F coincides with the convolution of H and F .

Remark 2.2. The co-monotonic case

In the case $C(w, \lambda) = w \wedge \lambda$, it is easy to verify

$$\begin{aligned} H \overset{C}{*} F(t) &= \int_0^1 \mathbb{I}_{(0, F(t - H^{-1}(w)))}(w) dw = \\ &= \int_0^1 \mathbb{I}_{\{w: F^{-1}(w) + H^{-1}(w) < t\}}(w) dw = \\ &= \sup \{w \in (0, 1) : H^{-1}(w) + F^{-1}(w) < t\} \end{aligned}$$

Remark 2.3. In the proof of Proposition 2.1, we chose to express the conditional distribution of Y with respect to X . Similarly, we could consider the distribution of X conditioned on Y . Hence, we can express $F_{X+Y}(t)$ in terms of $D_2 C(u, v) = \frac{\partial}{\partial v} C(u, v)$ as

$$F_{X+Y}(t) = \int_0^1 D_2 C(F_X(t - F_Y^{-1}(w)), w) dw$$

and all subsequent results be reformulated through $D_2 C$. In particular

$$F \overset{C}{*} G(t) = \int_0^1 D_1 C(w, G(t - F^{-1}(w))) dw = \int_0^1 D_2 C(F(t - G^{-1}(w)), w) dw \quad \forall t \quad a.e.$$

In the following we shall choose to represent the problem in terms of $D_1 C$, being the other choice the mirror image of that.

3 Sum with Bounded Support

Let us consider two random variables, X and Y . We want to investigate which conditions the c.d.fs F_X and F_Y and the copula C associated to the vector (X, Y)

have to satisfy in order to guarantee that the sum fulfils bounds constraints of type

$$\mathbb{P}(\beta \leq X + Y \leq \gamma) = 1 \quad (6)$$

where $\beta, \gamma \in \mathbb{R}$. Let us firstly consider the upper bound γ . The following result holds:

Theorem 3.1. *Let X and Y be two continuous random variables with c.d.f. F_X and F_Y and C the copula function associated to the random vector (X, Y) . Let us assume $D_1C(w, F_Y(\gamma - F_X^{-1}(w)))$ be well defined for all $w \in (0, 1)$ a.s.. $\mathbb{P}(X + Y \leq \gamma) = 1$ if and only if*

$$D_1C(w, F_Y(\gamma - F_X^{-1}(w))) = 1, \quad \forall w \in [0, 1] \quad a.s. \quad (7)$$

Proof. By (2)

$$F_{X+Y}(t) = \int_0^1 D_1C(w, F_Y(t - F_X^{-1}(w))) dw$$

and $\mathbb{P}(X + Y \leq \gamma) = 1$ if and only if

$$F_{X+Y}(\gamma) = \int_0^1 D_1C(w, F_Y(\gamma - F_X^{-1}(w))) dw = 1. \quad (8)$$

But $0 \leq D_1C(u, v) \leq 1$ (see Nelsen [?]) and (8) holds if and only if

$$D_1C(w, F_Y(\gamma - F_X^{-1}(w))) = 1, \quad \forall w \in [0, 1] \quad a.s.$$

□

Remark 3.1. *Since, by Lemma 2.1, $D_1C(t, F_Y(\gamma - F_X^{-1}(w)))$ is well defined for all w a.s. for all $t \geq 0$ a.e., if $D_1C(w, F_Y(\gamma - F_X^{-1}(w)))$ is not well defined for all $w \in (0, 1)$ a.s., then $\mathbb{P}(X + Y \leq \gamma) = 1$ if and only if there exists a sequence $\gamma_n \downarrow \gamma$ such that (7) is satisfied by γ_n for all n .*

Since of the above theorem, we are interested in studying the equation

$$D_1C(u, v) = 1. \quad (9)$$

Let $\mathcal{S}_C = \{(u, v) \in [0, 1]^2 : D_1C(u, v) = 1\}$.

Proposition 3.1. *The following statements hold:*

1. $(u, 1) \in \mathcal{S}_C$ for almost all $u \in [0, 1]$;
2. if $(u, v) \in \mathcal{S}_C$, $(u, v') \in \mathcal{S}_C$ for almost all $v' \geq v$;
3. $\forall u \in (0, 1)$, $(u, 0) \notin \mathcal{S}_C$.

Proof. 1. Since $C(u, 1) = u$, $D_1C(u, 1) = 1$ for almost all $u \in [0, 1]$.

2. Let $(u, v) \in \mathcal{S}_C$, that is $D_1C(u, v) = 1$. But $D_1C(u, v)$ is, for every given u , increasing with respect to v , (see [?]). This way, if $v' > v$, $D_1C(u, v) \leq D_1C(u, v')$. Moreover $0 \leq D_1C(u, v) \leq 1$ for almost all $(u, v) \in [0, 1]^2$ and so $D_1C(u, v') = 1$ when defined.
3. Since $C(u, 0) = 0$, this statement is obvious. □

Given a copula function C , (7) is satisfied if and only if F_X and F_Y define a no-increasing function $v(w) = F_Y(\gamma - F_X^{-1}(w))$ whose graph is contained in \mathcal{S}_C .

Let $\alpha_{max} = \inf\{x \in \mathbb{R} : F_X(x) \geq 1\}$ and $\beta_{max} = \inf\{x \in \mathbb{R} : F_Y(x) \geq 1\}$.

Proposition 3.2. *If $\alpha_{max} + \beta_{max} \leq \gamma$, (7) is satisfied for every copula function C and all marginal distributions F_X and F_Y .*

Proof. Since $F_X^{-1}(w) < \alpha_{max}, \forall w \in (0, 1), \gamma - F_X^{-1}(w) > \gamma - \alpha_{max} \geq \beta_{max}$. This way, it follows $F_Y(\gamma - F_X^{-1}(w)) = 1 \forall w$ a.s. and (7) holds by the first point of Proposition 3.1. □

Proposition 3.3. *If there exists an open interval $(a, b) \subset (0, 1)$ such that $\forall u \in (a, b)$, and $(u, v) \in \mathcal{S}_C$ if and only if $v = 1$, then (7) holds if and only if $\sup_{w \in (0, b)} F_X^{-1}(w) + \beta_{max} \leq \gamma$.*

Proof. For $w \in (a, b)$ a.s., $(\omega, v(\omega)) \in \mathcal{S}_C$ iff $v(\omega) = 1$ that is $\forall \omega \in (a, b), F_Y(\gamma - F_X^{-1}(w)) = 1$ or, equivalently, iff $\forall \omega \in (a, b), \gamma \geq F_X^{-1}(w) + \beta_{max}$. This is clearly equivalent to $\gamma \geq \sup_{w \in (a, b)} F_X^{-1}(w) + \beta_{max}$ and, being $w \rightarrow F_Y(\gamma - F_X^{-1}(w))$

a no increasing function, $\gamma \geq \sup_{w \in (0, b)} F_X^{-1}(w) + \beta_{max}$. □

Remark 3.2. *If we consider a bound from below that is $\mathbb{P}(X + Y \geq \beta) = 1$, thanks to Corollary 2.1, similarly we get $\mathbb{P}(X + Y \geq \beta) = 1 \Leftrightarrow D_1\overline{C}(\omega, \overline{F_Y}(\beta - \overline{F_X}^{-1}(\omega))) = 1$.*

Theorem 3.1 characterizes the request that the sum $X + Y$ is bounded from above, in terms of the derivative D_1C . It would be useful to translate this condition in terms of the copula function itself. This is the content of the following result:

Theorem 3.2. $\mathbb{P}(X + Y \leq \gamma) = 1$ if and only if $C(u, v) = u + v - 1$, for all $(u, v) \in A_\gamma$ a.s. where $A_\gamma = \{(u, v) \in [0, 1]^2 : v \geq F_Y(\gamma - F_X^{-1}(u))\}$. Moreover $F_Y(\gamma - F_X^{-1}(u)) \geq 1 - u$.

Proof. From Theorem 3.1

$$\mathbb{P}(X + Y \leq \gamma) = 1 \Leftrightarrow D_1 C(u, F_Y(\gamma - F_X^{-1}(u))) = 1 \quad \forall u \in [0, 1] \quad a.s.$$

But by Proposition 3.1 $D_1 C(u, v) = 1, \quad \forall (u, v)$ above the graph of the no increasing curve $(u, F_Y(\gamma - F_X^{-1}(u)))$, that is $D_1 C(u, v) = 1, \quad \forall (u, v) \in A_\gamma$ a.s.. This implies that $C(u, v) = u + h(v)$ and since $C(1, v) = v \Rightarrow 1 + h(v) = v \Rightarrow h(v) = v - 1$ and so $C(u, v) = u + v - 1, \quad \forall (u, v) \in A_\gamma$.

Moreover since for all copula functions $C(u, v) \geq 0$, to have $(u + v - 1)\mathbf{1}_{A_\gamma} \geq 0$ it must be $u + v - 1 \geq 0$ which implies $A_\gamma \subset \{(u, v) : v \geq 1 - u\}$ or $F_Y(\gamma - F_X^{-1}(u)) \geq 1 - u, \forall u. \quad \square$

Remark 3.3. *The content of previous Theorem is that C copula must coincide with minimum copula at least above the graph of $\bar{v}(u) = F_Y(\gamma - F_X^{-1}(u))$. But $\bar{v}(u)$ is a decreasing function. We then find the minimum set with decreasing lower boundary in which C coincides with the minimum copula.*

Let

$$\Phi = \{f : [0, 1] \rightarrow [0, 1] : f(u) \text{ decreasing and } C(u, v) = u + v - 1 \forall v \geq f(u)\}.$$

Let $f^*(u) = \inf_{f \in \Phi} f(u)$ (which is obviously decreasing as well). If $\epsilon > 0$ exists such that $\forall u \in (\epsilon, 1] f^*(u) = 1$, from Proposition 2.4 $\sup_{w \in (0, 1)} F_X^{-1}(w) + \beta_{max} \leq \gamma$ or $\alpha_{max} + \beta_{max} \leq \gamma$ (the trivial case in Proposition 2.3). We then rule out this trivial case.

Let then $M_C = \{(u, v) : v > f^*(u)\} \subset \mathcal{S}_C$.

Again, through Proposition 3.1, it is possible to express the bound from below in terms of the copula function C .

Theorem 3.3. $\mathbb{P}(X + Y \geq \beta) = 1$ if and only if $C(u, v) = 0$, for all $(u, v) \in A_\beta^C$ where $A_\beta = \{(u, v) \in [0, 1]^2 : v > F_Y(\beta - F_X^{-1}(u))\}$. Moreover $F_Y(\beta - F_X^{-1}(u)) \leq 1 - u$.

Proof. From Theorem 3.1

$$\mathbb{P}(X + Y \geq \beta) = 1 \Leftrightarrow D_1 \bar{C}(u, \bar{F}_Y(\beta - \bar{F}_X^{-1}(u))) = 1 \quad \forall u \in [0, 1] \quad a.s.$$

But

$$\begin{aligned} D_1 \bar{C}(u, \bar{F}_Y(\beta - \bar{F}_X^{-1}(u))) &= 1 - D_1 C(1 - u, F_Y(\beta - \bar{F}_X^{-1}(u))) = \\ &= 1 - D_1 C(1 - u, F_Y(\beta - F_X^{-1}(1 - u))) = 1 \\ &= \Leftrightarrow D_1 C(1 - u, F_Y(\beta - F_X^{-1}(u))) = 1 \\ &\Leftrightarrow D_1 C(u, F_Y(\beta - F_X^{-1}(u))) = 0 \end{aligned}$$

But being $D_1 C(u, v)$ non decreasing with respect to v , we have that $D_1 C(u, v) = 0, \quad \forall (u, v)$ under the graph of the curve $(u, F_Y(\beta - F_X^{-1}(u)))$ or $D_1 C(u, v) =$

0, $\forall (u, v) \in A_\beta^C$. This implies $C(u, v) = g(v)$ and since $C(0, v) = 0 \Rightarrow g(v) = 0 \Rightarrow C(u, v) = 0, \forall (u, v) \in A_\beta^C$.

Moreover because for all copula functions $C(u, v) \geq 0$, to have $(u+v-1)\mathbf{1}_{A_\beta} \geq 0$ it must be $u+v-1 \geq 0$ (see Theorem 3.2), then $A_\beta^C \subset \{(u, v) : v \leq 1-u\}$ or $F_Y(\beta - F_X^{-1}(u)) \leq 1-u, \forall u$. \square

Remark 3.4. *If we assume simultaneously the same bound from above and from below, that is $\mathbb{P}(X + Y = \gamma) = 1$ we have the following result;*

Corollary 3.4. $\mathbb{P}(X + Y = \gamma) = 1$ if and only if $C(u, v) = \max(u + v - 1, 0)$ and $F_Y(\gamma - F_X^{-1}(u)) = 1 - u$.

Proof. In this case, we must have

$$\begin{aligned} D_1 C(u, F_Y(\gamma - F_X^{-1}(u))) &= 1 \quad \forall u \in [0, 1] \quad a.s. \\ D_1 C(u, F_Y(\gamma - F_X^{-1}(u))) &= 0 \quad \forall u \in [0, 1] \quad a.s. \end{aligned}$$

which is impossible and which imply that the curve

$$(u, F_Y(\gamma - F_X^{-1}(u))) \in [0, 1] \times [0, 1]$$

belongs to the set of the points for which $D_1 C(u, v)$ does not exist. But by Theorem 3.2 and Theorem 3.3, $D_1 C(u, v) = 1, \forall (u, v)$ over the graph of the curve $(u, F_Y(\gamma - F_X^{-1}(u)))$, and $D_1 C(u, v) = 0, \forall (u, v)$ under the same curve and then $C(u, v) = (u + v - 1)\mathbf{1}_{A_\gamma}$.

Since $C(0, v) = 0$ for all copula functions, all the points $(0, v)$ must belong to A_γ^C this implies that $(0, F_Y(\gamma - F_X^{-1}(0))) = (0, 1)$ similarly since $C(1, v) = v$ for all copula functions, all the points $(1, v)$ must belong to A_γ , this implies that $(1, F_Y(\gamma - F_X^{-1}(1))) = (1, 0)$. Then the curve $(u, F_Y(\gamma - F_X^{-1}(u)))$ has extremes points in $(0, 1)$ and $(1, 0)$.

Moreover by Theorem 3.2 $F_Y(\gamma - F_X^{-1}(u)) \geq 1 - u, \forall u$. But if it exists \hat{u} such that $F_Y(\gamma - F_X^{-1}(\hat{u})) > 1 - \hat{u}$, so we would have $C(u, v) = (u + v - 1)\mathbf{1}_{A_\gamma} \leq \max(u + v - 1, 0)$ with the strict inequality on the points (\hat{u}, v) such that $1 - \hat{u} < v \leq F_Y(\gamma - F_X^{-1}(\hat{u}))$. Being this impossible, it must be $F_Y(\gamma - F_X^{-1}(u)) = 1 - u$ and $C(u, v) = \max(u + v - 1, 0)$. \square

3.1 Examples

Proposition 3.4. *Let $C(u, v)$ be the Marshall-Olkin copula with parameters m and n :*

$$C(u, v) = vu^{1-m}\mathbb{I}_{\{u^m \geq v^n\}} + uv^{1-n}\mathbb{I}_{\{u^m < v^n\}}$$

Then $\mathcal{S}_C = \{(u, 1) : u \in [0, 1]\}$.

Proof. Obviously

$$D_1 C(u, v) = (1 - m)u^{-m}v\mathbb{I}_{\{u^m > v^n\}} + v^{1-n}\mathbb{I}_{\{u^m < v^n\}}.$$

(9) becomes

$$(1 - m)u^{-m}v\mathbb{I}_{\{u^m > v^n\}} + v^{1-n}\mathbb{I}_{\{u^m < v^n\}} = 1. \quad (10)$$

Notice that both addends must be 1. But, from

$$(1 - m)u^{-m}v = 1,$$

$$v = \frac{u^m}{(1 - m)}.$$

If $u^m > v^n$, that is if $(1 - m)v > v^n$, $v^{1-n} > \frac{1}{1-m}$ which is impossible being $\frac{1}{1-m} > 1$. This way, only the second addend is well defined and

$$v^{1-n} = 1$$

from which

$$v = 1, \forall u \text{ a.e.}$$

□

Proposition 3.5. *Let $C(u, v)$ be an Archimedean copula with generator ϕ :*

$$C(u, v) = \phi^{-1}(\phi(u) + \phi(v))$$

where ϕ^{-1} is the pseudo-inverse function of ϕ .

We have that $\mathcal{S}_C = \{(u, 1) : u \in [0, 1]\}$ if and only if ϕ' is invertible.

If ϕ' is not invertible there exists an open interval (a, b) such that

$$\{(u, v) : u \in (a, b), v \in (\phi^{-1}(\phi(a) - \phi(u)), 1]\} \subset \mathcal{S}_C.$$

Proof. Let us start with the strict copula case, that is the case $\phi(0) = +\infty$

$$D_1 C(u, v) = \frac{\phi'(u)}{\phi'(\phi^{-1}(\phi(u) + \phi(v)))}.$$

Hence by (9)

$$\frac{\phi'(u)}{\phi'(\phi^{-1}(\phi(u) + \phi(v)))} = 1$$

and

$$\phi'(u) = \phi'(\phi^{-1}(\phi(u) + \phi(v))).$$

Being ϕ' invertible

$$u = \phi^{-1}(\phi(u) + \phi(v))$$

from which

$$\phi(u) = \phi(u) + \phi(v).$$

Hence

$$\phi(v) = 0$$

that is $v = 1$.

Let us now consider the case $\phi(0) < +\infty$.

Let $A = \{(u, v) \in [0, 1]^2 : \phi(u) + \phi(v) > \phi(0)\}$. By definition

$$C(u, v) = \begin{cases} \phi^{-1}(\phi(u) + \phi(v)), & \text{if } (u, v) \in A \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $D_1 C(u, v) = 0$ for all $(u, v) \in A^c$. This way only pairs $(u, v) \in A$ are candidate to satisfy (9).

But, for every $u \in (0, 1)$, let $A(u) = \{v \in [0, 1] : \phi(u) + \phi(v) \leq \phi(0)\}$. $A(u)$ is not empty, since $\phi(u) + \phi(v) \leq \phi(0)$ implies $\phi(v) \leq \phi(0) - \phi(u)$, that is $v \geq \phi^{-1}(\phi(0) - \phi(u))$. But $0 < \phi(0) - \phi(u) < \phi(0)$, that is $0 < \phi^{-1}(\phi(0) - \phi(u)) < 1$, which imply that $A(u)$ is not empty.

Now, for every $u \in (0, 1)$ and $v \in A(u)$ all arguments showed in the case $\phi(0) = +\infty$ hold again, and so the only solution is $v = 1$.

On the other hand, let u_1 and u_2 such that $u_1 < u_2$ and $\phi'(u_1) = \phi'(u_2) = m$. Being ϕ convex, this implies $\phi'(u) = m$ for almost every $u \in [u_1, u_2]$. Let (a, b) be the maximal open interval such that $\phi'(u) = m$ for almost every $u \in (a, b)$. Now, for almost every $u \in (a, b)$ condition (9) is satisfied for a certain $v \in A(u)$ (in the case $\phi(0) = +\infty$, $A(u) = [0, 1]$), if and only if

$$\phi'(u) = \phi'(C(u, v))$$

that is if and only if $C(u, v) \in (a, b)$. This implies

$$a < \phi^{-1}(\phi(u) + \phi(v)) < b$$

that is

$$\phi(b) - \phi(u) < \phi(v) < \phi(a) - \phi(u).$$

But $\phi(b) - \phi(u) \leq 0$, this way we obtain

$$0 \leq \phi(v) < \phi(a) - \phi(u).$$

thai is

$$\phi^{-1}(\phi(a) - \phi(u)) < v \leq 1.$$

Notice that $(\phi^{-1}(\phi(a) - \phi(u)), 1] \subset A(u)$ and $\phi^{-1}(\phi(a) - \phi(u)) = 1$ if and only if $u = a$. This way, for every $u \in (a, b)$ there exists a non empty interval $(\phi^{-1}(\phi(a) - \phi(u)), 1]$ of solutions $v(u)$. \square

Moreover in this case it's possible to identify the level set $\phi^{-1}(\phi(a) - \phi(u))$. Let $\hat{v}(u) = \phi^{-1}(\phi(a) - \phi(u))$, $\forall u \in (a, b)$. Since the generator ϕ is strictly decreasing, \hat{v} is decreasing as well. Since $\hat{v}(a) = 1$, $\hat{v}(b) < 1$. To rule out the trivial case above we must necessarily consider $b = 1$. Since \hat{v} is decreasing

$$N_C = \{(u, v) : v \geq \hat{v}(u)\} \subset M_C$$

where we remind that $M_C = \{(u, v) : v > f^*(u)\} \subset \mathcal{S}_C$ so that $C(u, v) = u+v-1$ $\forall (u, v) \in N_C$. Since however

$$C(u, \hat{v}(u)) = a$$

it necessarily follows that this must be the a -level curve of the minimum copula, then $\hat{v}(u) = \phi^{-1}(\phi(a) - \phi(u)) = -u + a + 1$ for all $u \in [a, 1]$.

Remark 3.5. The lower Fréchet bound.

In this case $\phi(u) = 1 - u$ and

$$C(u, v) = (u + v - 1)\mathbb{I}_{\{u+v-1 \geq 0\}}.$$

Furthermore, the interval $[a, b]$ of Proposition 3.5 is $[0, 1]$ and the interval of all solutions $[\phi^{-1}(\phi(a) - \phi(u)), 1]$ is $[\phi^{-1}(\phi(0) - \phi(u)), 1] = [\phi^{-1}(u), 1] = [1 - u, 1]$. This way $\mathcal{S}_C = \{(u, v) \in [0, 1]^2 : v \geq 1 - u\}$.

Proposition 3.6. *Let $C(u, v) = \sum_{i=1}^n \lambda_i C^i(u, v)$, $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$, then $\mathcal{S}_C = \bigcap_{i=1}^n \mathcal{S}_{C^i}$.*

Proof. Since $D_1 C(u, v) = \sum_{i=1}^n \lambda_i D_1 C^i(u, v)$ and being $0 \leq D_1 C^i(u, v) \leq 1 \forall i$, $\forall (u, v) \in (0, 1)$ a.s., $D_1 C(u, v) = 1 \Leftrightarrow D_1 C^i(u, v) = 1 \forall i = 1, \dots, n$. \square

Proposition 3.7. *Let $C_\rho(u, v)$ be a Gaussian copula:*

$$C_\rho(u, v) = \int_0^u \phi \left(\frac{\phi^{-1}(v) - \rho \phi^{-1}(t)}{\sqrt{1 - \rho^2}} \right) dt$$

then $\mathcal{S}_{C_\rho} = \{(u, v) : v = 1, u \in (0, 1)\}$.

Proof. The condition to verify is $D_1 C_\rho(u, v) = \phi \left(\frac{\phi^{-1}(v) - \rho \phi^{-1}(u)}{\sqrt{1 - \rho^2}} \right) = 1$, but $\forall (u, v) \in (0, 1)$ this is impossible and the thesis follows by the first part of Proposition 3.1. \square

4 Discrete Random Variables

In this section we focus on the distribution of the sum of two discrete random variables. Firstly we study the general unbounded setting and then we consider the boundness constraints.

4.1 Unbounded case

Proposition 4.1. *Let X and Y be two discrete random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $C(\omega, \lambda)$ be any copula function associated to (X, Y) . If $\{x_j\}_{j \in I_X}$, with I_X almost countable, is the set of values taken by X with positive probabilities, we assume $x_j \leq x_{j+1} \forall j \in I_X$. Let $F_X(x_j) = p_j$, then*

$$F_{X+Y}(t) = \sum_{j \in I_X} [C(p_j, F_Y(t - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(t - F_X^{-1}(p_j)))]. \quad (11)$$

Proof.

$$\begin{aligned} F_{X, X+Y}(s, t) &= \sum_{x_j \leq s} \mathbb{P}(X + Y \leq t, X = x_j) = \\ &= \sum_{x_j \leq s} \mathbb{P}(Y \leq t - x_j, X = x_j) = \\ &= \sum_{x_j \leq s} [\mathbb{P}(X \leq x_j, Y \leq t - x_j) - \mathbb{P}(X \leq x_{j-1}, Y \leq t - x_j)] = \\ &= \sum_{x_j \leq s} [C(F_X(x_j), F_Y(t - x_j)) - C(F_X(x_{j-1}), F_Y(t - x_j))] \\ &= \sum_{p_j \leq F_X(s)} [C(p_j, F_Y(t - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(t - F_X^{-1}(p_j)))] \end{aligned}$$

As $s \uparrow +\infty$, we obtain (11). □

Remark 4.1. *We can observe that if $F_X(s) = u$, $F_{X+Y}(t) = v$, then trivially follows*

$$C_{X, X+Y}(u, v) = \sum_{p_j \leq u} [C(p_j, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(p_j)))] \quad (12)$$

Corollary 4.1. *Let X and Y be two discrete random variables on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $C(\omega, \lambda)$ be any copula function associated to (X, Y) . If $\{x_j\}_{j \in I_X}$, with I_X almost countable, is the set of values taken by X with positive probabilities, we assume $x_j \leq x_{j+1} \forall j \in I_X$. Let $F_X(x_j) = p_j$, then*

$$\bar{F}_{X+Y}(t) = \sum_{j \in I_X} [\bar{C}(1 - p_{j-1}, \bar{F}_Y(t - F_X^{-1}(p_j))) - \bar{C}(p_j, \bar{F}_Y(t - F_X^{-1}(p_j)))] \quad (13)$$

Proof.

$$\begin{aligned}
\bar{F}_{X+Y}(t) &= 1 - \sum_{j \in I_X} [\bar{C}(1 - p_j, \bar{F}_Y(t - F_X^{-1}(p_j))) - (1 - p_j + \bar{F}_Y(t - F_X^{-1}(p_j))) + \\
&\quad - \bar{C}(p_{j-1}, \bar{F}_Y(t - F_X^{-1}(p_j))) + 1 - p_{j-1} + \bar{F}_Y(t - F_X^{-1}(p_j))] \\
&= 1 - \sum_{j \in I_X} [\bar{C}(1 - p_j, \bar{F}_Y(t - F_X^{-1}(p_j))) - \bar{C}(p_{j-1}, \bar{F}_Y(t - F_X^{-1}(p_j))) + p_j - p_{j-1}] \\
&= \sum_{j \in I_X} [\bar{C}(1 - p_j, \bar{F}_Y(t - F_X^{-1}(p_j))) - \bar{C}(p_{j-1}, \bar{F}_Y(t - F_X^{-1}(p_j)))]
\end{aligned}$$

□

4.2 Sum with bounded support

In the continuous case, the boundness constraints was firstly written in terms of the first partial derivative of the copula function C and, after it was expressed in terms of the copula function C itself. On the other hand, in the discrete setting, we recover immediately a condition on the copula function C .

Proposition 4.2. *In the same setting of Proposition 4.1, $\mathbb{P}(X + Y \leq \gamma)$ iff $\forall j \in I_X$*

$$p_j - p_{j-1} = C(p_j, F_Y(\gamma - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(\gamma - F_X^{-1}(p_j))) \quad (14)$$

Proof. By (11) we have

$$\begin{aligned}
F_{X+Y}(\gamma) &= \sum_{j \in I_X} [C(p_j, F_Y(\gamma - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(\gamma - F_X^{-1}(p_j)))] = \\
&= \sum_{j \in I_X} \mathbb{P}(p_{j-1} < U \leq p_j, V \leq F_Y(\gamma - F_X^{-1}(p_j))) \leq \\
&\leq \sum_{j \in I_X} \mathbb{P}(p_{j-1} < U \leq p_j) = \sum_{j \in I_X} (p_j - p_{j-1}) = 1
\end{aligned}$$

Hence $F_{X+Y}(\gamma) = 1$ iff $\forall j \in I_X$

$$C(p_j, F_Y(\gamma - F_X^{-1}(p_j))) - C(p_{j-1}, F_Y(\gamma - F_X^{-1}(p_j))) = p_j - p_{j-1}$$

□

From Proposition 4.2

$$C(u_j, F_Y(\gamma - F_X^{-1}(u_j))) - C(u_{j-1}, F_Y(\gamma - F_X^{-1}(u_j))) = u_j - u_{j-1}$$

or

$$C(u_j, v_j) - C(u_{j-1}, v_j) = u_j - u_{j-1} \quad (15)$$

Let \mathcal{S}_C^D be

$$\mathcal{S}_C^D = \{(u_j, v_j)\}_{j \in I} : u_j \text{ is strictly increasing and (15) is satisfied}\}$$

Proposition 4.3. *The following statements hold:*

1. $\forall \{u_j\}_{j \in I}, \{(u_j, 1)\}_{j \in I} \in \mathcal{S}_C^D$;
2. if $\{(u_j, v_j)\}_{j \in I} \in \mathcal{S}_C^D, \{(u_j, v'_j)\}_{j \in I} \in \mathcal{S}_C^D, \forall v'_j \geq v_j$;
3. $\forall \{u_j\}_{j \in I}, \{(u_j, 0)\}_{j \in I} \notin \mathcal{S}_C^D$.

Proof. As the proof of Proposition 3.1. □

The statement of Proposition 3.2 and Proposition 3.3 can be proved identically in the discrete setting;

Proposition 4.4. *If $\alpha_{\max} + \beta_{\max} \leq \gamma$, (14) is satisfied for every copula function C and all marginal distributions F_X and F_Y .*

Proposition 4.5. *Let $\{u_j\}_{j \in I}$ be with values in $[0, 1]$ and strictly increasing. If there exists $J \subseteq I$ such that $\{(u_j, v_j)\}_{j \in I} \in \mathcal{S}_C^D$ iff $v_j = 1, \forall j \in J$, then (14) holds iff $\sup_{j \in J} F_X^{-1}(u_j) + \beta_{\max} \leq \gamma$.*

Proof. The proof is analogous to the one of Proposition 3.3. □

Remark 4.2. *If we consider a bound from below that is $\mathbb{P}(X + Y \geq \beta) = 1$, thanks to Corollary 4.1, similarly we get*

$$\mathbb{P}(X+Y \geq \beta) = 1 \Leftrightarrow \forall j \in I_X, \bar{C}(1-p_j, \bar{F}_Y(t-F_X^{-1}(p_j))) - \bar{C}(p_{j-1}, \bar{F}_Y(t-F_X^{-1}(p_j))) = p_j - p_{j-1}$$

5 An Application to Reinsurance

In this section we present an application to actuarial science. Assume that an insurer has exposures to two different sources of risk, and may face losses X and Y , stemming from each risk factor. The dependence structure of losses X and Y is described by a copula function $C(u, v)$. In the example at hand, and just for the sake of illustration, we take the product copula, so that the two losses are independent. The joint distribution is described in the contour plot in Figure 1 representing the set of combinations of probability of losses consistent with an assigned level of default probability. In risk management, the contour plot can be used to allocate capital between one loss and the other.

INSERT FIGURE 1 ABOUT HERE

Now assume that the portfolio is reinsured against overall losses higher than amount γ . We may either assume that the reinsurer is default free or that reinsurance is sought by setting aside an amount γ in cash or risk-free assets. Then, losses will be bounded above by the amount γ with probability 1. We will have

$$X + Y \leq \gamma \text{ a.s.}$$

We are interested in the question of how the contour plot, and the dependence structure of the losses, are changed under the reinsurance policy. Figure 2 reports the new contour plot taking into account the constraint on losses implied by the reinsurance decision. We observe that the level curves tend to flatten out as we get closer and closer to the level where

$$D_1C(u, v) = D_2C(u, v) = 1$$

INSERT FIGURE 2 ABOUT HERE

For Archimedean copulas, an alternative representation of the change in the dependence structure is given by the point beyond which the generator becomes linear. In our specific case of independent random variables we have $\phi(t) = -\ln(t)$. The change induced in the dependence structure by the reinsurance decision is represented in Figure 3. We see that the reinsurance policy introduces a kink in the generator, and this kink corresponds to the value at which the level curves become straight lines. This is the point when the constraint becomes binding, meaning that the reinsurance policy on losses comes into effect.

INSERT FIGURE 3 ABOUT HERE

In order to give a comparison of the dependence structure of losses with and without reinsurance in Figure 4 we report the contour plots in the two cases.

INSERT FIGURE 4 ABOUT HERE

6 Conclusions

The distribution of sum of independent random variables is obtained by means of the convolution of their distribution. In this paper, we explore the extension of this concept to the case in which the random variables in the sum are not independent. In order to address the problem in full generality we represent dependence using copula functions. The extension provides a fully general technique to compute the distribution of sums of random variables. As for computability, a numerical algorithm for the evaluation of convolutions of dependent variables is presented in Cherubini, Mulinacci and Romagnoli (2008) and applied to the credit analysis of a portfolio. In this paper, we extend the analysis to the case in which the sum of the variables is bounded. We show that this introduces constraints in the partial derivatives of copula functions, representing conditional distributions. We show that some classes of copula functions are not consistent with these constraints, and therefore cannot be used to represent the dependence structure of two variables whose sum is bounded. In particular, neither Marshall-Olkin or elliptical copulas can be used in these cases (unless the trivial one in which the sum of the upper bounds of the supports is lower than the upper bound of the sum). As for Archimedean copulas, we show that if the sum of the variables is bounded, there must exist a region in which the derivative of the generator is not invertible, and this implies that it must be

linear in the same region. We also show that in that region the copula becomes the minimum copula $u + v - 1$, corresponding to perfect negative dependence between the variables. The level curves in that region are therefore downward sloping straight lines. As an example of application, we study the change of dependence structure between two risks once a reinsurance policy is introduced. We find that the contour plots of the level curves, that are typically used for the allocation of capital, are considerably flattened out.

References

- [1] Cherubini U., Mulinacci S. and S. Romagnoli (2008): "A Copula-Based Model of the Term Structure of CDO Tranches", in Hardle W.K., N. Hautsch and L. Overbeck (eds) *Applied Quantitative Finance*, Springer Verlag, Heidelberg 69-81.
- [2] Denuit, M., Genest, C. and É. Marceau, (1999): "Stochastic bounds on sums of dependent risks", *Insurance: Mathematics and Economics*, 25, 85-104;
- [3] Frank, M.J., Nelsen, R. and B. Schweizer, (1987): "Best-possible distributions of a sum - a problem of Kolmogorov", *Prob. Th. Rel. Fields*, 74, 199-211;
- [4] Makarov, G.D., (1981): "Estimates for the distribution function of a sum of two random variables when the marginal distributions are fixed", *Theory Prob. Appl.*, 26, 803-806;
- [5] Moynihan, R., Schweizer, B. and A. Sklar (1978): "Inequalities among binary operations on probability distribution functions", *General Inequalities*, 1, 133-149;

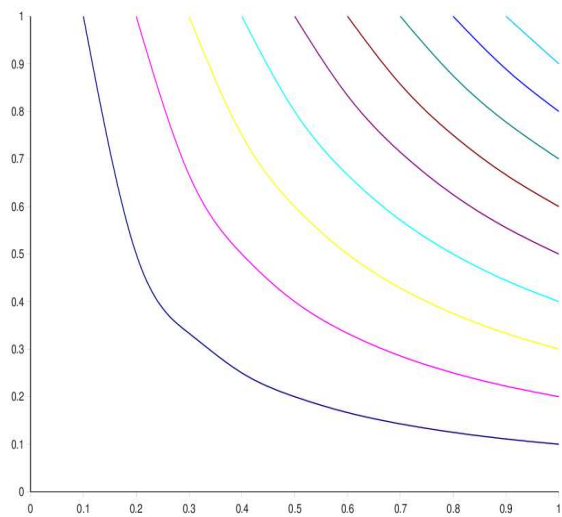


Figure 1: Contour plot of losses X and Y without reinsurance

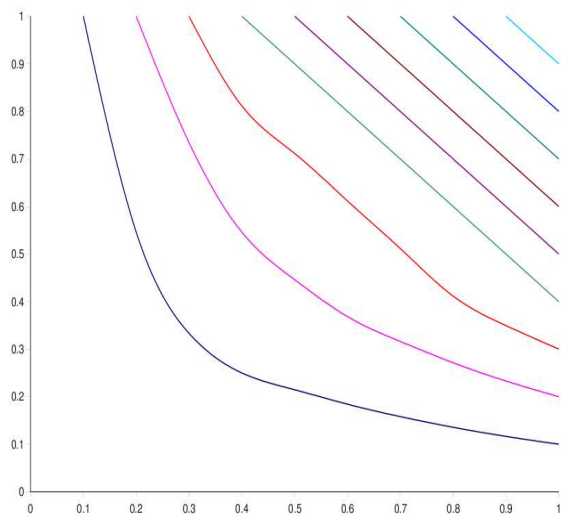


Figure 2: Contour plot of losses X and Y with reinsurance

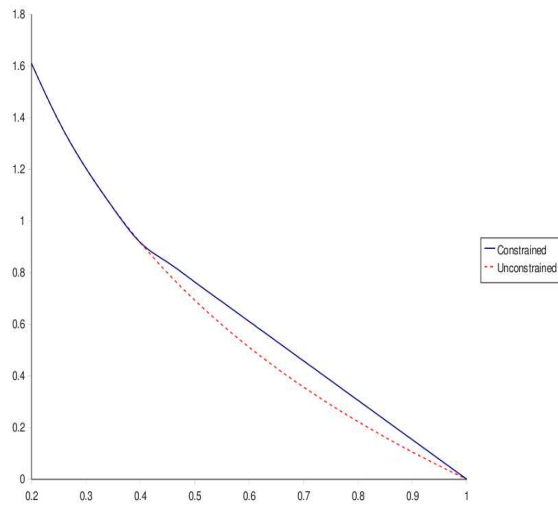


Figure 3: Generator of the copula function linking losses X and Y with and without reinsurance

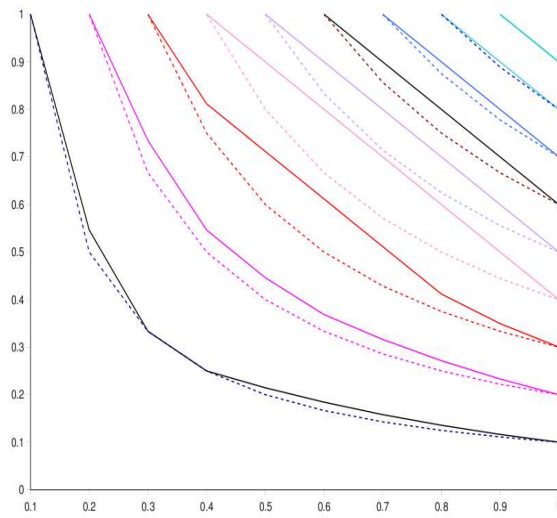


Figure 4: Comparison of contour plots of losses X and Y with and without reinsurance