On the Term Structure of Multivariate Equity Derivatives *

Umberto Cherubini † Fabio Gobbi ‡ Sabrina Mulinacci§
Silvia Romagnoli ¶

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Abstract

We present a copula-based technique to recover the price relationship of multivariate equity derivatives expiring at different dates, consistently with the martingale requirement. We derive the entire term structure of equity derivatives in a SCOMDY model with univariate marginal Markov processes and no-Granger causality. We show that this structure allows to model the univariate dynamics of each asset before the multivariate dynamics and to impose the martingale requirement on each of them. We then prove that the martingale requirement is preserved under enlargement of the filtration to the history of all the assets in the basket. In this arbitrage-free framework, we provide recursions for evaluation of the term structure of i) multivariate pricing kernels; ii) correlation. The recursion is particularly easy to carry out in the case in which the multivariate log-price process is with independent increments, or can be turned out into such process by means of change of time. We provide pricing surfaces of multivariate digital products (Altiplanos), rainbow options (call on min notes, or Everest) and spread options.

Keywords: Option pricing, Markov processes, Granger causality, copula functions.

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†University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094370; Fax: +(39) 0512094357. e-mail: umberto.cherubini@unibo.it

‡University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094368; Fax: +(39) 0512094357. e-mail: fabio.gobbi@unibo.it

§University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094368; Fax: +(39) 0512094357. e-mail: sabrina.mulinacci@unibo.it

¶University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy. Phone: +(39) 0512094340; Fax: +(39) 0512094357. e-mail: silvia.romagnoli@unibo.it

1 Introduction

Assume you observe, for some asset S^j, a set of option prices for different time horizons t_i. From these options, one can extract information on the term structure of the distribution of that asset. By the same token, if one was able to observe a set of multivariate option prices on a basket of assets with different maturities, she could extract information about the joint distribution of those assets. Unfortunately, in standard applications of copula functions to option pricing this temporal dimension is lost. In these models, equity basket derivative prices are obtained by applying copulas to marginal distributions calibrated for each given maturity independently of the others. Whether these prices are consistent across different maturities remains on open question, and this is the topic that we are going to address in this paper.
Behind the need to preserve temporal consistency there is a much more substantial requirement, that has to do with enforcing the martingale condition, that makes the pricing model arbitrage-free. Of course, this condition is required to hold under the filtration generated by all the assets involved in the price of the derivative contracts, and this makes the problem even harder to solve.
Heuristically, one could think of modelling the dynamics of every asset at the univariate level, making sure that the martingale requirement is satisfied at that level, and finally linking the univariate distributions with a copula function to obtain the joint distribution at each time t_i. Would this procedure preserve the martingale property at the multivariate level? Or it is merely an approximation of the correct price that one could obtain by directly enforcing the martingale condition under the filtration generated by the prices of all the assets? Or else, is there a market model under which the copula pricing approach would yield results consistent with martingale pricing? And finally, if that is the case, would the procedure work for all copulas, or would it imply restrictions on the copula functions used? All of these are the questions addressed in this paper.
Notice that the issue behind these questions, whether a set of processes which are martingales with respect the filtration generated by each of them can preserve the martingale property under the enlarged filtration generated by all of them, is known in the literature as the *H-condition*. So, if the *H-condition* holds, the term structure of a multivariate equity derivative can be constructed by first modelling the term structures of univariate derivatives, and then applying copulas: this is what we call *bottom-up* approach. If such condition does not hold, then a more general approach would be needed: borrowing the term from the basket credit derivatives literature, we may call this approach *top-down*. This approach would require to impose the martingale restriction directly under the enlarged filtration generated by the prices of all the assets, and the univariate term structures would be obtained as a by-product of the multivariate one.

The result of our analysis, that we anticipate here, is that the *bottom-up* approach would yield consistent results in a market model in which all prices (log-prices, to be precise) of assets are generated by a multivariate Markov process with no-Granger causality. The no-Granger causality condition is defined as the case in which the prediction of future values of a variable based on its

own past history cannot be improved by adding the history of any of the others. This condition is pretty weak since it corresponds to the requirement of *market efficiency in semistrong form* that is commonly used in standard asset pricing models. This is good news because it means that the flexibility of copula functions is preserved under the very basic assumption of market dynamics implied by the market efficiency hypothesis, that is that future movements of the assets cannot be predicted.
The good news has to be balanced against a bad one, or at least a warning: the copula functions that are used must be consistent with the independent increment structure of this market model. So, for example, if you assume that the multivariate distribution evolves with temporally independent increments which are jointly described by an Archimedean structure that remains constant through time, this would not result in an Archimedean dependence structure among prices, and not even to a constant dependence structure across maturities. Even though the assumption of independent increments would enforce the conditions required for a *bottom-up* approach, the dependence structure among levels could only be simulated even in this simplest framework.

The paper is organized as follows. In section 2 we describe top-down versus bottom-up approaches to the evaluation of multivariate equity derivatives in this setting. In section 3 we describe the main requirements of a market model in discrete time inspired to the SCOMDY architecture in which a copula pricing model could be applied. In section 4 we show how to enforce the martingale requirement in such model. In section 5 we we derive recursively the copula functions among the level of the variables in the SCOMDY model. In section 6 we specialize the recursion to the case in which the multivariate process is with independent increments, or can be turned into such process by a suitable change of time. In section 7 we build independent increment models by applying a suitable change of time. In section 8 we add a recursion of correlation at different time horizons in the same framework. In section 9 we present simulations for the pricing surface (over different time horizons and strikes) of multivariate digital products (Altiplanos) call options on the minimum of two assets (Everest) and spread options. Section 10 concludes.

2 Top-Down vs Bottom-Up Pricing

Consider now the problem of pricing a multivariate derivative. We may consider two different approaches.

Definition 2.1. ***Top-Down Pricing****: the price of each and every derivative is computed under the assumption that the prices of the underlying assets be martingale processes with respect to the filtration generated by all assets.*

This approach is very general and can be applied to whatever dynamics of the prices in the system. Every price is assumed to be a martingale with respect to all available information. This would imply that univariate claims would be martingale as well, and could be used to price univariate derivatives. The term *top-down* is used to stress the feature that arbitrage-free univariate prices are obtained as marginals of arbitrage free multivariate prices. The difficulty with this approach is to come out with a model that could satisfactorily fit marginal distributions. The pros of this approach is that in one single shot it provides arbitrage free prices for both univariate and multivariate products. The cons is that the multivariate model used may be the wrong one, leaving substantial room for model risk in the process.

The first models for the evaluation of multivariate equity derivatives are of the *top-down* kind. In their pioneering work, Margrabe (1978) and Johnson (1986) provided prices for exchange options and rainbow options under the assumption of multivariate Black and Scholes setting with constant volatility and correlation. More recently Driessen et al. (2005) provided an example of modern *top-down approach* to multivariate option pricing with time changing correlation. Their basic idea is to assume normally distributed returns conditional on the correlation. As for the dynamics of correlation, they provide a mean reverting Jacobi process. A natural extension of this idea was to model the entire multivariate dynamics of the correlation matrix. In the probability literature this problem had been first tackled by Bru (1991) who proposed a Wishart dynamics for the correlation matrix. Da Fonseca et al. (2007) proposed to apply this model to equity multivariate derivatives, and also designed restrictions to make the model affine in the risk factors. The most recent proposal of a *top-down approach* is due to Carr and Lawrence (2010), that provide multivariate extensions of the Breeden and Litenberger (1978) and Dupire (1996) methodologies to fit the smile and term structure surfaces. This approach can be considered the most flexible among the *top-down* ones, since it aims at building a multivariate model consistent with the local volatility surface of all the assets involved.

Definition 2.2. ***Bottom-Up Pricing****: the dynamics of each and every asset is first modelled as a univariate martingale process with respect to its own filtration, and copula functions C_t are then applied to recover the joint distribution of prices at each point in time t. The prices of multivariate equity derivatives are computed using this joint distribution.*

In *bottom-up* pricing, arbitrage free marginal prices are computed in the first place, and then linked using copula functions to yield multivariate prices. The pros of the approach is that it provides a direct link to the prices of the assets, which is the only source of information that one is able to collect on financial markets. The cons is that the model is intrinsically static, and it is not possible, in principle, to compare multivariate prices for different maturities. Neither is it possible to provide dynamic hedging strategies. This flaw is twofold. On one side, correlation is not modelled as a stochastic process. On the other, there is no guarantee that the dynamics of asset prices is modelled in an arbitrage free setting. The first issue can be easily addressed by resorting to dynamic copula models, namely the concept of conditional copula due to Patton (2001) and to that of pseudo-copula proposed by Fermanian and Wegkamp (2004). The second issue is much more involved because it requires that the forward prices

be martingale under the measure used for pricing. In general the requirement that each marginal price be a martingale with respect to its own filtration does not imply that it remains a martingale with respect to the enlarged filtration generated by the history of all asset prices. If this occurs, we say that the system satisfies the *H-condition*. So, only if the *H-condition* holds the univariate arbitrage free prices can be aggregated to generate a multivariate arbitrage free pricing system. Whenever the condition does not hold, the *bottom-up* pricing procedure would leave room for arbitrage possibilities.
The first proposals of *bottom-up* pricing models go back to Rosenberg (2003) and Cherubini and Luciano (2002). Van den Goorbergh et al. (2005) extended the model to allow for time varying dependence. Cherubini and Romagnoli (2009) explored the scalability of the model with respect to dimension and the put-call parity relationship. In a most recent paper Cherubini and Romagnoli (2010) provided the first attempt to cast the model in a dynamics perspective, proposing a *bootstrap* approach based on copula functions to extract the term structure of barrier multivariate derivatives. Luciano and Schoutens (2006) propose a model in which the copula based approach is mixed with a time change structure generated by a common stochastic clock which provides the model with a *top-down* flavor.

3 The Market Model

We now describe the market price dynamics of asset returns. The model is in discrete time and is specified in a semi-parametric fashion in order to separately represent univariate dynamics and the dependence structure. The reference in econometrics is the so-called SCOMDY (Semiparametric Copula-based Multivariate Dynamics) proposed by Chen and Fan (2006).

Formally, we assume a filtered probability space $\{\mathbf{\Omega}, \mathcal{F}_t, \mathbb{P}\}$ satisfying the usual conditions. As usual, we work with logarithm of prices. So, if S is the price of an asset, we deal with X, defined as

$$S = \exp(X)$$

The model is in discrete time, so we consider a set of periods limited by dates $\{0, t_1, t_2, ..., t_n\}$. The setting is multivariate, so that we denote with X_i^j, $j = 1, 2, ..., m$, the log-prices of assets in the economy at time t_i. As for the modelling strategy, we focus attention on the log-price increments of each asset X^j, namely: $\left\{Y_1^j, Y_2^j, ..., Y_n^j\right\}$. Obviously, we have $X_i^j = X_{i-1}^j + Y_i^j$.

The multivariate structure requires further specification of the information structure that will turn out paramount in the development of the analysis. Namely, we denote by $\mathcal{F}_t^j$ the filtration collecting the information generated by the history of asset j. The overall filtration $\mathcal{F}_t$ embeds the pieces of information generated by all the assets.

We are now ready we describe the main assumptions of the market model driving the dynamics of assets:

- Assumption 1 - The logarithm of each price is a Markov process with respect to its natural filtration.
- Assumption 2 - Each asset is not Granger-caused by any of the others.

It is now clear that the hypotheses above describe the standard framework of the *Efficient Market Hypothesis.* Neither information about the history of a single variable or that of the others can possibly help to forecast the future development of the market. More precisely, Assumption 1 provides a definition of what is known as *weak-form efficiency.* Assumption 2 extends the concept to the more strict criterium of market efficiency in the *semi-strong form.*
Within the framework just described, we may now introduce the main requirement of arbitrage free pricing.

- Assumption 3 - There exists a probability measure under which the forward price of each asset is a martingale with respect to its natural filtration.

Let us remark again that, given the multivariate setting, the proper specification of the probability space and the information structure requires that the no arbitrage prices must reflect all the available information. More explicitly, all assets must be martingale with respect to filtration $\mathcal{F}_t$.

4 The Martingale Property

We now show that under the market model described in section (3) the *H-condition holds*, and the *bottom-up* and *top-down* approach would yield consistent results. We construct this proof by: i) constructing marginal distributions endowed with the martingale property; ii) proving that under the no-Granger causality hypothesis a Markov process with respect to a filtration remains a Markov process with respect to the larger filtration; iii) showing that this result allows to exploit previous findings on the *H-condition* stating that in this case the martingale property is preserved in the larger filtration.

4.1 Marginal Martingale Processes

For each asset we assume that the log-price increment Y_i^j be endowed with a probability distribution $F_{Y_i^j}$. Likewise, we denote $F_{X_i^j}$ the set of distributions of log-prices. Of course, we have $F_{Y_1^j} = F_{X_1^j}$. We also assume a set of copula functions $C_{X_{i-1}^j, Y_i^j}$ representing the dependence structure between the value of the asset at the beginning of the period and its increment in that period.

Given the Markovian structure of the process, we could use the Cherubini, Mulinacci and Romagnoli (2009) approach to recover the distribution of X_i^j from that of X_{i-i}^j and Y_i^j. The relationship is given by an extension of the concept of convolution,

$$F_{X_i^j}(t) = \int_0^1 D_1 C_{X_{i-1}^j, Y_i^j}\left(w, F_{Y_i^j}(t - F^{-1}_{X_{i-1}^j}(w))\right) dw. \quad (1)$$

By the same token, the temporal dependence structure of the process, that is the dependence between X_{i-1}^j and X_i^j is given by the copula function

$$G_{X_{i-1}^j, X_i^j}(u,v) = \int_0^u D_1 C_{X_{i-1}^j, Y_i^j}\left(w, F_{Y_i^j}(F^{-1}_{X_i^j}(v) - F^{-1}_{X_{i-1}^j}(w)\right) dw. \quad (2)$$

We now show how to impose the martingale restriction in this setting. It would be easy to derive the result under the assumption of independent increments. Since however this restriction may turn out to be much too coarse for some applications, we derive the result in the most general setting that allows for dependence of the distribution of each increment on the previous level value.

More specifically, in what follows, we device a fully general procedure to construct a sequence of log-increment distributions and a sequence of copula functions describing the dependence structure between each log-price level and the next log-increment so that the resulting asset prices process is a martingale.

Since we are referring here to the natural filtration generated by every log-price X_i^j, for the ease of notation we drop superscript j. The idea is to recursively define a compensated process $\hat{X}_i$ satisfying the martingale requirement starting from a given sequence of increments Y_i. The recursive definition is

$$H(\hat{X}_{n-1}) = \ln(\mathbb{E}[e^{Y_n}|\hat{X}_{n-1}])$$
$$Z_n = Y_n - H(\hat{X}_{n-1})$$
$$\hat{X}_n = \hat{X}_{n-1} + Z_n$$
$$S_n = e^{\hat{X}_n}$$

with the initial condition $\hat{X}_0 = 0$. In plain terms, we define a series of *compensated* increments Z_i and the corresponding series of compensated log-prices $\hat{X}_i$ by integration. We assume that each Y_n is independent of $\mathcal{F}^{\hat{X}}_{n-1} = \sigma(\hat{X}_i : i \leq n-1)$ conditioned on $\hat{X}_{n-1}$, that is, formally

$$\mathbb{P}(Y_n \leq x|\mathcal{F}^{\hat{X}}_{n-1}) = \mathbb{P}(Y_n \leq x|\hat{X}_{n-1}).$$

This implies that the same property holds for Z_n; in fact

$$\mathbb{P}(Z_n \leq x|\mathcal{F}^{\hat{X}}_{n-1}) = \mathbb{P}(Y_n - H(\hat{X}_{n-1}) \leq x|\mathcal{F}^{\hat{X}}_{n-1}) =$$
$$= \mathbb{P}(Y_n - H(\hat{X}_{n-1}) \leq x|\hat{X}_{n-1}) = \mathbb{P}(Z_n \leq x|\hat{X}_{n-1}).$$

It is easy to show that the stochastic process S_n is a Markov process, satisfying *Assunmption 1*

$$\mathbb{P}(S_n \leq x|\mathcal{F}^S_{n-1}) = \mathbb{P}(S_{n-1}e^{Z_n} \leq x|\mathcal{F}^S_{n-1}) =$$

$$= \mathbb{P}(S_{n-1}e^{Z_n} \le x | S_{n-1}) = \mathbb{P}(S_{n-1}e^{Z_n} \le x | \hat{X}_{n-1}).$$

Moreover S_n is a martingale, as we require:

$$\begin{aligned}
\mathbb{E}[S_n|\hat{X}_{n-1}] &= \mathbb{E}[S_{n-1}e^{Z_n}|\hat{X}_{n-1}] = \\
&= S_{n-1}\mathbb{E}[e^{Z_n}|\hat{X}_{n-1}] = \\
&= S_{n-1}\mathbb{E}[e^{Y_n - H(\hat{X}_{n-1})}|\hat{X}_{n-1}] = \\
&= S_{n-1}\mathbb{E}[e^{Y_n}|\hat{X}_{n-1}]e^{-H(\hat{X}_{n-1})} = \\
&= S_{n-1}\mathbb{E}[e^{Y_n}|\hat{X}_{n-1}](\mathbb{E}[e^{Y_n}|\hat{X}_{n-1}])^{-1} = \\
&= S_{n-1}
\end{aligned}$$

Notice that in this new setting, F_{Z_n} cannot be arbitrarily chosen since Z_n is by definition a *compensated* process and it depends on the copula linking $\hat{X}_{n-1}$ and Y_n through $\mathbb{E}[e^{Y_n}|\hat{X}_{n-1}]$. Moreover, given the copula $T_{\hat{X}_{n-1},Y_n}$, also the copula $T_{\hat{X}_n,Z_n}$ depends on it. But it is not hard work to compute F_{Z_n} and $T_{\hat{X}_n,Z_n}$ starting from F_{Y_n} and $T_{\hat{X}_n,Y_n}$.

Let us start with the joint distribution $F_{\hat{X}_{n-1},Z_n}$

$$\begin{aligned}
F_{\hat{X}_{n-1},Z_n}(x,z) &= \mathbb{P}(\hat{X}_{n-1} \le x, Z_n \le z) = \\
&= \mathbb{P}(\hat{X}_{n-1} \le x, Y_n - H(\hat{X}_{n-1}) \le z) = \\
&= \int_{-\infty}^{x} \mathbb{P}(Y_n - H(t) \le z | \hat{X}_{n-1} = t) dF_{X_{n-1}}(t) = \\
&= \int_{-\infty}^{x} \mathbb{P}(Y_n \le z + H(t) | \hat{X}_{n-1} = t) dF_{X_{n-1}}(t) = \\
&= \int_{-\infty}^{x} D_1 T_{\hat{X}_{n-1},Y_n}(F_{\hat{X}_{n-1}}(t), F_{Y_n}(z + H(t))) dF_{X_{n-1}}(t) = \\
&= \int_{0}^{F_{\hat{X}_{n-1}}(x)} D_1 T_{\hat{X}_{n-1},Y_n}(w, F_{Y_n}(z + H(F^{-1}_{\hat{X}_{n-1}}(w)))) dw
\end{aligned}$$

From this we can obtain F_{Z_n} by simply letting $x \to +\infty$:

$$F_{Z_n}(z) = \int_0^1 D_1 T_{\hat{X}_{n-1},Y_n}(w, F_{Y_n}(z + H(F^{-1}_{\hat{X}_{n-1}}(w)))) dw$$

and, by Sklar theorem, the copula $T_{\hat{X}_{n-1},Z_n}$ is

$$T_{\hat{X}_{n-1},Z_n}(u,v) = \int_0^u D_1 T_{\hat{X}_{n-1},Y_n}(w, F_{Y_n}(F^{-1}_{Z_n}(v) + H(F^{-1}_{\hat{X}_{n-1}}(w)))) dw$$

Now, we have all the ingredients to start our algorithm to compute the distribution of $\hat{X}_n$ and the copula function of $(\hat{X}_{n-1}, \hat{X}_n)$ by applying (1) and (2).

Remark 4.1. *Constructing martingale processes for the marginals would be much easier under the assumption of independent increments. In this case all one would need is to normalize the increments*

$$S_i^j = \frac{e^{X_i^j}}{\mathbb{E}(e^{X_i^j})}.$$

and it is very easy to verify that S_i^j *is a martingale. Even though the assumption of independent increments may be questioned on the basis that it may be difficult to generate the skew observed on the market at finite maturities, in most of the models commonly in use one can generally transform the process into one with independent increments under a suitable change of time. This route was actually followed by Cherubini and Romagnoli (2010) to derive a recursion for univariate discrete barrier options.*

4.2 Multivariate Martingale Processes

When pricing multivariate claims, the martingale restriction must be defined with respect to the enlarged filtration containing information generated by all the assets in the basket. While the main advantage of *top-down* models is that this martingale requirement is included from the very start, in the *bottom-up* approach based on copulas this condition should be impounded in the model once that the martingale marginal processes have been specified. Luckily, we are going to show that the no-Granger causality *Assumption 2* is sufficient to enforce this requirement. Furthermore, it is a hypothesis that is almost always empirically verified in financial markets.

In econometrics, no-Granger-causality means that no infomation can help to predict the future values of a variable over and above the past history of the variable itself. In other words, all information needed to represent the probability distribution of a variable is contained in the history of the variable itself.

Definition 4.1. $X^1, \ldots, X^{j-1}, X^{j+1}, \ldots, X^m$ *do not Granger cause* X^j *if*

$$\mathbb{P}[X_{k+1}^j \leq x | \mathcal{F}_k] = \mathbb{P}[X_{k+1}^j \leq x | \mathcal{F}_k^{X^j}]$$

for any time t_k *and* x.

Remember that a process that is Markov with respect to a given filtration, is not in general Markov with respect to a larger filtration. We now show that this is in fact guaranteed by no-Granger causality.

Theorem 4.2. *The following are equivalent:*

1. X^j *is not Granger caused by* $X^1, \ldots, X^{j-1}, X^{j+1}, \ldots, X^m$;
2. *if* X^j *is an* $\mathcal{F}_k^{X^j}$*-Markov process, then it is an* $\mathcal{F}_k$*-Markov process, as well.*

Proof. 1. ⇒ 2.:1. implies

$$\mathbb{P}[X^j_{k+1} \leq x | \mathcal{F}_k] = \mathbb{P}[X^j_{k+1} \leq x | \mathcal{F}^{X^j}_k].$$

for every $x \in \mathbb{R}$. By hypothesis

$$\mathbb{P}[X^j_{k+1} \leq x | \mathcal{F}^{X^j}_k] = \mathbb{P}[X^j_{k+1} \leq x | X^j_k]$$

and the thesis follows. The other implication is trivial. □

Based on this theorem, it is easy to show that the *H-condition* holds for the whole Markov system. We recall in fact that the concept of no-causality allows to ensure the stability of the martingale property with respect to enlarged filtrations (see Florens and Fugère, 1991). More specifically, this is a consequence of the above theorem and of the fact that: if S^j is both an $\mathcal{F}^{X^j}$ and an $\mathcal{F}$-Markov process and it is furthermore an $\mathcal{F}^{X^i}$-martingale, it turns out to be an $\mathcal{F}$-martingale as well (see Brémaud and Yor, 1978).

Remark 4.2. *Notice that the no-Granger causality condition implies restrictions on the copula functions that can be used to represent the dynamics of the Markov process. A study on these restrictions may be found in Cherubini, Mulinacci and Romagnoli (2009).*

5 SCOMDY market model: copula recursion

We now describe in more detail how to recover the dependence structure of the variables recursively in a SCOMDY model. While the result goes through in a very general setting, it can be used to provide a general recursion for arbitrage free prices of multivariate derivatives if we assume that the system satisfies the martingale conditions above.

We report the general result in the following theorem, that is spelled out in a bivariate setting for the sake of simplifying notation.

Theorem 5.1. *Assume two variables X^j and X^k generated by the model*

$$X^j_i = X^j_{i-1} + \epsilon_i$$

$$X^k_i = X^k_{i-1} + \eta_i$$

with ϵ_i and η_i described by marginal distributions F_{ϵ_i} and F_{η_i}, with dependence structure given by copulas

$$C_i(u, v, u', v') = C_{X^j_{i-1}, X^k_{i-1}, \epsilon_i, \eta_i}(u, v, u', v').$$

We assume that the second partial derivatives with respect to the first two arguments of this copulas exist and to simplify the notation we set

$$C_{i|X^j_{i-1}, X^k_{i-1}}(u, v, u', v') = \frac{\frac{\partial^2 C_i(u,v,u',v')}{\partial u \partial v}}{\frac{\partial^2 C_i(u,v,1,1)}{\partial u \partial v}}.$$

Then, the copula function describing dependence between X_i^j and X_i^k is given by the recursion

$$C_{X_i^j,X_i^k}(u,v) =$$

$$\int_0^1\int_0^1 C_{i|X_{i-1}^j,X_{i-1}^k}\Big(\omega,\varsigma,F_{\varepsilon_i}\big(F^{-1}_{X_i^j}(u)-F^{-1}_{X_{i-1}^j}(\omega)\big),F_{\eta_i}\big(F^{-1}_{X_i^k}(v)-F^{-1}_{X_{i-1}^k}(\varsigma)\big)\Big)dC_{X_{i-1}^j,X_{i-1}^k}(\omega,\varsigma).$$

where the distribution functions of X_i^j and X_i^k are given by the generalized convolution equations

$$F_{X_i^j}(x) = \int_0^1 D_1C_{X_{i-1}^j,\varepsilon_i}\left(\omega,F_{\varepsilon_i}(x-F^{-1}_{X_{i-1}^j}(\omega))\right)d\omega$$

and

$$F_{X_i^k}(z) = \int_0^1 D_1C_{X_{i-1}^k,\eta_i}\left(\omega,F_{\eta_i}(z-F^{-1}_{X_{i-1}^k}(\omega))\right)d\omega.$$

Proof. We have

$$F_{X_i^j,X_i^k}(x,z) = \mathbb{P}(X_i^j\le x,X_i^k\le z) =$$

$$\mathbb{P}(X_{i-1}^j+\varepsilon_i\le x,X_{i-1}^k+\eta_i\le z) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{P}(X_{i-1}^j+\varepsilon_i\le x,X_{i-1}^k+\eta_i\le z|X_{i-1}^j=s,X_{i-1}^k=t)dF_{X_{i-1}^j,X_{i-1}^k}(s,t) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{P}(\varepsilon_i\le x-s,\eta_i\le z-t|X_{i-1}^j=s,X_{i-1}^k=t)dF_{X_{i-1}^j,X_{i-1}^k}(s,t) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} C_{i|X_{i-1}^j,X_{i-1}^k}\big(F_{X_{i-1}^j}(s),F_{X_{i-1}^k}(t),F_{\varepsilon_i}(x-s),F_{\eta_i}(z-t)\big)dC_{X_{i-1}^j,X_{i-1}^k}\big(F_{X_{i-1}^j}(s),F_{X_{i-1}^k}(t)\big).$$

Therefore the copula linking X_i^j and X_i^k is given by

$$C_{X_i^j,X_i^k}(u,v) = F_{X_i^j,X_i^k}\big(F^{-1}_{X_i^j}(u),F^{-1}_{X_i^k}(v)\big) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} C_{i|X_{i-1}^j,X_{i-1}^k}\big(F_{X_{i-1}^j}(s),F_{X_{i-1}^k}(t),F_{\varepsilon_i}(F^{-1}_{X_i^j}(u)-s),F_{\eta_i}(F^{-1}_{X_i^k}(v)-t)\big)dC_{X_{i-1}^j,X_{i-1}^k}\big(F_{X_{i-1}^j}(s),F_{X_{i-1}^k}(t)\big).$$

With the change of variables $\omega = F_{X_{i-1}^j}(s)$ and $\varsigma = F_{X_{i-1}^k}(t)$, we get

$$C_{X_i^j,X_i^k}(u,v) =$$

$$\int_0^1\int_0^1 C_{i|X_{i-1}^j,X_{i-1}^k}\Big(\omega,\varsigma,F_{\varepsilon_i}\big(F^{-1}_{X_i^j}(u)-F^{-1}_{X_{i-1}^j}(\omega)\big),F_{\eta_i}\big(F^{-1}_{X_i^k}(v)-F^{-1}_{X_{i-1}^k}(\varsigma)\big)\Big)dC_{X_{i-1}^j,X_{i-1}^k}(\omega,\varsigma).$$

As for the distribution functions of X_i^j and X_i^k, the convolution structure of the marginal distribution derives immediately from (1). □

Example 5.1. ***Altiplanos***. *The recursion above can be used to enforce no-arbitrage relationships among multivariate digital derivatives of increasing maturities. Namely, starting with the dependence structure of increments for the first option, we iteratively compute the options of the following maturities. Of course, the distribution of the increments must satisfy the martingale requirements in Section 4.*

Remark 5.1. *Notice that in the case in which the vector of the increments are independent the result above is simplified and the copula linking X_i^j and X_i^k becomes*

$$C_{X_i^j, X_i^k}(u, v) =$$

$$\int_0^1 \int_0^1 C_i\Big(F_{\varepsilon_i}\big(F^{-1}_{X_i^j}(u) - F^{-1}_{X_{i-1}^j}(\omega)\big), F_{\eta_i}\big(F^{-1}_{X_i^k}(v) - F^{-1}_{X_{i-1}^k}(\varsigma)\big)\Big) dC_{X_{i-1}^j, X_{i-1}^k}(\omega, \varsigma)$$

where $C_i = C_{\epsilon_i, \eta_i}$.

6 Componentwise, Vector and Granger Independent Increments

In this section we provide a discussion of the implementation of our model under the hypothesis of independent increments. The discussion will be also linked to the cases in which the log-price processes can be transformed into independent increment processes by means of a change of time. This instance will be addressed explicitly in the following section.We start by noticing that in a multivariate setting the term *independent increments* may indicate different concepts, as described in the definitions below.

Definition 6.1. *We say that a set of Markov processes $X = (X^1, \ldots, X^m)$ are* componentwise *independent if for any k*

$$\mathbb{P}(\Delta X_i^k \le x | X_{i-1}^k) = \mathbb{P}(\Delta X_i^k \le x)$$

where $\Delta X_i^k = X_i^k - X_{i-1}^k$.

Definition 6.2. *We say that a set of Markov processes $X = (X^1, \ldots, X^m)$ are* vector *independent if*

$$\mathbb{P}(\Delta X_i^1 \le x_1, \ldots, \Delta X_i^m \le x_m | X_{i-1}^1, \ldots, X_{i-1}^m) = \mathbb{P}(\Delta X_i^1 \le x_1, \ldots, \Delta X_i^m \le x_m).$$

Obviously, if a set of processes is vector independent it is componentwise independent as well, but the opposite is not true. Our market model is somewhat in the middle between these two extreme cases and uses a concept of independence that we define *Granger independence*.

Definition 6.3. *We say that a set of processes $X = (X^1, \dots, X^m)$ are* Granger *independent if for any k*

$$\mathbb{P}(\Delta X_i^k \le x | X_{i-1}^1, \dots, X_{i-1}^m) = \mathbb{P}(\Delta X_i^k \le x).$$

It is easy to check that the concept of Granger independence is somewhat the sum of the assumptions of componentwise independence and no-Granger causality. Formally

Proposition 6.1. *A set of processes which is componentwise independent and that satisfies the no-Granger causality assumption is Granger independent.*

Proof. The proof trivially follows from Definitions 4.1 and 6.1. □

While at first sight it may seem that vector independence and Granger independence are actually very similar, we may easily prove that they would lead to different prices for multivariate derivatives if the dependence structure of the increments changes in time. The following example provides a clear and neat intuition of this effect.

Example 6.1. *Consider two assets X_i^j and X_i^k evolving according to*

$$X_i^j = X_{i-1}^j + \epsilon_i$$

$$X_i^k = X_{i-1}^k + \eta_i$$

with $X_0^j = X_0^k = 0$ and a three period model with four states $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ at the final time t_2. The information structure evolves in such a way as to reveal at time t_1 whether the system will evolve towards the states $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$. For the sake of simplicity we assume the shocks to be $0-1$ and perfectly dependent at time t_1:

$$\epsilon_1 = \eta_1 = \mathbf{1}_{\{\omega_1, \omega_2\}}.$$

As for time t_2, we assume them to be perfectly correlated if the states $\{\omega_1, \omega_2\}$ occur and perfectly negatively correlated in the other states

$$\epsilon_2 = \mathbf{1}_{\{\omega_1, \omega_3\}},\ \eta_2 = \mathbf{1}_{\{\omega_1, \omega_4\}}.$$

In fact, if at time t_1 the system is in $\{\omega_1, \omega_2\}$ we have:

$$\epsilon_2 = \eta_2 = \mathbf{1}_{\{\omega_1\}}$$

while in the case $\{\omega_3, \omega_4\}$ the increments dynamics is

$$\epsilon_2 = 1 - \eta_2 = \mathbf{1}_{\{\omega_3\}}.$$

Now assume that the probabilities attached to each state of nature $\omega_i, i = 1, \dots, 4$ are equal to $1/4$. The probabilities attached to the set $\{\omega_1, \omega_2\}$, to $\{\omega_1\}$ conditional on observing $\{\omega_1, \omega_2\}$ at time t_1 and to $\{\omega_3\}$ conditional on observing $\{\omega_3, \omega_4\}$ are all equal to $1/2$. It may be proved that this structure is consistent *with Granger independence, but not with vector independence. In fact, consider the price of a claim paying one unit of cash if $\epsilon_1 = \eta_1 = \epsilon_2 = \eta_2 = 1$. It will be worth*

$$\Pr(\epsilon_1 = \eta_1 = \epsilon_2 = \eta_2 = 1) = \Pr(\epsilon_2 = \eta_2 = 1 | \epsilon_1 = \eta_1 = 1) \Pr(\epsilon_1 = \eta_1 = 1) = 0.25$$

Instead, assuming vector independence would lead to the wrong price:

$$\Pr(\epsilon_1 = \eta_1 = \epsilon_2 = \eta_2 = 1) = \Pr(\epsilon_2 = \eta_2 = 1) \Pr(\epsilon_1 = \eta_1 = 1) = 0.125$$

An interesting question is how the term structure of Altiplanos is affected by Granger-independent or vector independent increments. The answer is easily recovered on the basis of the following definition, which extends the concept of *stochastic increasing* relationship (see Joe, 2007) to the multivariate setting.

Definition 6.4. *A vector of variables $\boldsymbol{u}$ is said to be vector stochastic increasing (decreasing) in the vector of variables $\boldsymbol{v}$ if the conditional distribution*

$$\Pr(u_1 \leq x_1, u_2 \leq x_2, \ldots, u_n \leq x_n | \boldsymbol{v} = \boldsymbol{y})$$

is increasing (decreasing) in $\boldsymbol{v}$.

Clearly, from the above definition it immediately turns out that assuming vector independence leads to undervaluation of the pricing kernel in case the price increments of each period are stochastic increasing in those of the previous one.

7 Conditionally Independent Increments

We now show how to build independent increment models by applying a suitable time change. We refer to these cases as *conditionally independent* increment processes.

In this general setting, we consider a multidimensional stochastic process $X = (X^1, \ldots X^m)$ with stationary and vector independent increments and a multidimensional positive and increasing stochastic process $T = (T^1, \ldots T^m)$ such that $T_0 = (0, \ldots, 0)$ representing the stochastic clocks set. We assume T and X to be independent.

Lemma 7.1. *$X_{T_t} - X_{T_s}$ and $X_{T_t - T_s}$ have the same distributions.*

Proof. Let $\mathbf{v} \in \mathbb{R}^m$. We want to show that

$$\phi_{X_{T_t} - X_{T_s}}(\mathbf{v}) = \phi_{X_{T_t - T_s}}(\mathbf{v})$$

where by $\phi_X(v)$ we denote the characteristic function of X.

$$\begin{aligned}
\phi_{X_{T_t}-X_{T_s}}(\mathbf{v}) &= \mathbb{E}[e^{i\mathbf{v}\cdot(X_{T_t}-X_{T_s})}] = \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot(X_{r+h}-X_r)}]dF_{T_s,T_t-T_s}(r,h) = \text{ by stationarity} \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot(X_h)}]dF_{T_s,T_t-T_s}(r,h) = \\
&= \int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot(X_h)}]dF_{T_t-T_s}(h) = \\
&= \mathbb{E}[e^{i\mathbf{v}\cdot(X_{T_t-T_s})}] = \phi_{X_{T_t-T_s}}(\mathbf{v})
\end{aligned}$$

□

Proposition 7.1. *Let $X = (X^1, \dots, X^m)$ be a vector of processes whose increments are stationary and vector independent. Let $T = (T^1, \dots, T^m)$ be a multidimenstional stochastic process which is increasing and such that $T_0 = (T_0^1, \dots, T_0^m) = (0, \dots, 0)$. We assume T vector independent as well. Then The multidimensional time-changed process $X_T = (X^1_{T^1}, \dots, X^m_{T^m})$ is vector independent.*

Proof. We want to show that for all $\mathbf{v}, \mathbf{u} \in \mathbb{R}^m$

$$\phi_{(X_{T_t}-X_{T_s}, X_{T_s})}((\mathbf{v}, \mathbf{u})) = \phi_{X_{T_t}-X_{T_s}}(\mathbf{v})\phi_{X_{T_s}}(\mathbf{u}).$$

$$\begin{aligned}
\phi_{(X_{T_t}-X_{T_s}, X_{T_s})}((\mathbf{v}, \mathbf{u})) &= \mathbb{E}[e^{i(\mathbf{v},\mathbf{u})\cdot(X_{T_t}-X_{T_s}, X_{T_s})}] = \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i(\mathbf{v},\mathbf{u})\cdot(X_{r+h}-X_r, X_r)}]dF_{T_s,T_t-T_s}(r,h) = \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot(X_{r+h}-X_r)}]\mathbb{E}[e^{i\mathbf{u}\cdot X_r}]dF_{T_s,T_t-T_s}(r,h) = \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot X_h}]\mathbb{E}[e^{i\mathbf{u}\cdot X_r}]dF_{T_s,T_t-T_s}(r,h) = \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot X_h}]\mathbb{E}[e^{i\mathbf{u}\cdot X_r}]dF_{T_s}(r)dF_{T_t-T_s}(h) = \\
&= \int_0^{+\infty} \mathbb{E}[e^{i\mathbf{v}\cdot X_h}]dF_{T_t-T_s}(h)\int_0^{+\infty} \mathbb{E}[e^{i\mathbf{u}\cdot X_r}]dF_{T_s}(r) = \\
&= \mathbb{E}[e^{i\mathbf{v}\cdot(X_{T_t-T_s})}]\mathbb{E}[e^{i\mathbf{u}\cdot X_{T_s}}] = \text{ by Lemma 7.1} \\
&= \mathbb{E}[e^{i\mathbf{v}\cdot(X_{T_t}-X_{T_s})}]\mathbb{E}[e^{i\mathbf{u}\cdot X_{T_s}}]
\end{aligned}$$

□

Proposition 7.2. *Let $X = (X^1, \dots, X^m)$ be a vector of processes whose increments are stationary and vector independent. Let $T = (T^1, \dots, T^m)$ be a multidimenstional stochastic process which is increasing and such that $T_0 = (T_0^1, \dots, T_0^m) = (0, \dots, 0)$. We assume T Granger independent. Then The* *multidimensional time-changed process* $X_T = (X^1_{T^1}, \dots, X^m_{T^m})$ *is Granger independent as well.*

Proof. We want to show that for all $j = 1, \dots, m$, $\mathbf{u} \in \mathbb{R}^m$, $v \in \mathbb{R}$

$$\phi_{(X^j_{T_t} - X^j_{T_s}, X_{T_s})}((v, \mathbf{u})) = \phi_{X^j_{T_t} - X^j_{T_s}}(v\phi_{X_{T_s}}(\mathbf{u}).$$

This can be trivially obtained following the same steps as in the previous proof. □

From now on we are going to consider the case of a stochastic clock that is a Markov process with respect to its natural filtration. We will show that, under suitable but very general assumptions, it is possible to construct univariate martingales in the "economic" setting so that the resulting time changed processes are martingales with respect to the filtration generated by the entire multivariate time changed market model.
More precisely, let us consider a multidimensional stochastic process $X = (X^1, \dots X^m)$ with Granger independent increments and a multidimensional positive and increasing stochastic process $T = (T^1, \dots T^m)$ such that $T_0 = (0, \dots, 0)$ representing the stochastic clocks set. The stochastic process T is a multidimensional Markov process with respect to its natural filtration, meaning that for all $t > t_n > \dots > t_1 \geq 0$

$$\mathbb{P}(T_t \leq \mathbf{v} | T_{t_n} = \mathbf{v}_n, \dots, T_{t_1} = \mathbf{v}_1) = \mathbb{P}(T_t \leq \mathbf{v} | T_{t_n} = \mathbf{v}_n)$$

We assume T and X to be independent.

Lemma 7.2. *The above setting given, we have*

$$\mathbb{P}(X^j_{T^j_t} - X^j_{T^j_s} \leq x | \mathcal{F}^{X_T, T}_s) = \mathbb{P}(X^j_{T^j_t} - X^j_{T^j_s} \leq x | T_s) \tag{3}$$

where $\mathcal{F}^{X_T, T}_s$ *is the filtration generated by the* $2m$*-dimensional stochastic process* (X_{T_t}, T_t).

Proof. We have

$$\mathbb{P}(X^j_{T^j_t} - X^j_{T^j_s} \leq x | X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, X_{T_{s_{n-1}}} = \mathbf{x}_{n-1}, T_{s_{n-1}} = \mathbf{t}_{n-1}, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1) =$$
$$= \int_{t^j_n}^{+\infty} \mathbb{P}(X^j_v - X^j_{t^j_n} \leq x | T^j_t = v, X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1)$$
$$d\mathbb{P}(T^j_t \leq v | X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1) =$$
$$= \int_{t^j_n}^{+\infty} \mathbb{P}(X^j_v - X^j_{t^j_n} \leq x) d\mathbb{P}(T^j_t \leq v | X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1)$$

But

$$\begin{aligned}
&\mathbb{P}(T_t^j \le v | X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1) = \\
&= \frac{\mathbb{P}(T_t^j \le v, X_{\mathbf{t}_n} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{\mathbf{t}_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1)}{\mathbb{P}(X_{\mathbf{t}_n} = \mathbf{x}_n, T_s = \mathbf{t}_n, \dots X_{T_{\mathbf{t}_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1)} = \\
&= \frac{\mathbb{P}(T_t^j \le v, T_s = \mathbf{t}_n, \dots, T_{s_1} = \mathbf{t}_1)}{\mathbb{P}(T_s = \mathbf{t}_n, \dots, T_{s_1} = \mathbf{t}_1)} = \\
&= \mathbb{P}(T_t^j \le v | T_s = \mathbf{t}_n, \dots, T_{s_1} = \mathbf{t}_1) = \\
&= \mathbb{P}(T_t^j \le v | T_s = \mathbf{t}_n)
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{P}(X^j_{T_t^j} - X^j_{T_s^j} \le x | X_{T_s} = \mathbf{x}_n, T_s = \mathbf{t}_n, X_{T_{s_{n-1}}} = \mathbf{x}_{n-1}, T_{s_{n-1}} = \mathbf{t}_{n-1}, \dots X_{T_{s_1}} = \mathbf{x}_1, T_{s_1} = \mathbf{t}_1) = \\
&= \int_{t_n^j}^{+\infty} \mathbb{P}(X_v^j - X^j_{t_n^j} \le x) d\mathbb{P}(T_t^j \le v | T_s = \mathbf{t}_n) = \\
&= \mathbb{P}(X^j_{T_t^j} - X^j_{T_s^j} \le x | T_s = \mathbf{t}_n).
\end{aligned}$$

□

Proposition 7.3. *In the same set of assumptions stated above, we have that, if $Z_t^j = e^{X_t^j}$ is a martingale with respect to its natural filtration $\mathcal{F}_t^{X^j}$, then the time changed stochastic process $S_t^j = e^{X^j_{T_t^j}}$ is a martingale with respect to the filtration $\mathcal{F}_t^{X_T}$ generated by the multidimensional time changed stochastic process $S = (e^{X^1_{T^1}}, e^{X^2_{T^2}}, \dots, e^{X^m_{T^m}})$.*

Proof. By the hypotheses on X and S, we have that for all $u > v$, $\mathbb{E}\left[e^{X_u^j - X_v^j}\right] = 1$. Now

$$\begin{aligned}
\mathbb{E}\left[S_t^j - S_s^j | \mathcal{F}_s^{X_T,T}\right] &= S_s^j \mathbb{E}\left[e^{X^j_{T_t^j} - X^j_{T_s^j}} - 1 | \mathcal{F}_s^{X_T,T}\right] = \text{ by the above Lemma} \\
&= S_s^j \mathbb{E}\left[e^{X^j_{T_t^j} - X^j_{T_s^j}} - 1 | T_s\right]
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\left[e^{X^j_{T_t^j} - X^j_{T_s^j}} - 1 | T_s = v\right] &= \int_v^{+\infty} \mathbb{E}\left[e^{X_u^j - X_v^j} - 1 | T_t = u, T_s = v\right] d\mathbb{P}(T_t \le u | T_s = v) = \\
&= \int_v^{+\infty} \mathbb{E}\left[e^{X_u^j - X_v^j} - 1\right] d\mathbb{P}(T_t \le u | T_s = v) = 0
\end{aligned}$$

This shows that each S^j is a martingale with respect to the filtration $\mathcal{F}_t^{X_T,T}$; but then each S^j is a martingale with respect to the smaller filtration $\mathcal{F}_t^{X_T}$ as required.

□

8 SCOMDY market model: correlation recursion

In case that one is not interested in the evolution of the whole joint distribution over different maturities, but only in the evolution of correlation, we now show how to recover a simple recursion of correlation at different maturities in the SCOMDY model above. The recursion is again valid under increment independence or conditional independence.

We denote $\mu_{\epsilon,i}$ and $\mu_{\eta,i}$ the mean of the innovations ϵ and η at time t_i, by $\sigma_{\epsilon,i}$ and $\sigma_{\eta,i}$ their volatility and by ρ_i their correlation coefficient.

By independence we have

$$\mathbb{E}[X_i^j] = \mu_{X_0^j} + \Sigma_{m=1}^i \mu_{\epsilon,m}, \quad Var(X_i^j) := \sigma^2_{X_i^j} = \sigma^2_{X_0^j} + \Sigma_{m=1}^i \sigma^2_{\epsilon,m},$$

$$\mathbb{E}[X_i^k] = \mu_{X_0^k} + \Sigma_{m=1}^i \mu_{\eta,m}, \quad Var(X_i^k) := \sigma^2_{X_i^k} = \sigma^2_{X_0^k} + \Sigma_{m=1}^i \sigma^2_{\epsilon,m}.$$

We compute the correlation coefficient between log-prices X_i^j and X_i^k. We have

$$\rho_{X_i^j, X_i^k} = \frac{\mathbb{E}[X_i^j X_i^k] - \mathbb{E}[X_i^j]\mathbb{E}[X_i^k]}{\sigma_{X_i^j}\sigma_{X_i^k}}.$$

Since

$$\mathbb{E}[X_i^j X_i^k] =$$
$$\mathbb{E}[X_{i-1}^j X_{i-1}^k] + \mathbb{E}[X_{i-1}^j]\mathbb{E}[\eta_i] + \mathbb{E}[X_{i-1}^k]\mathbb{E}[\varepsilon_i] + \mathbb{E}[\varepsilon_i \eta_i],$$

and

$$\mathbb{E}[\varepsilon_i \eta_i] = \rho_i \sigma_{\varepsilon,i}\sigma_{\eta,i} + \mu_{\varepsilon,i}\mu_{\eta,i},$$

after some algebra

$$\rho_{X_i^j, X_i^k} = \frac{\mathbb{E}[X_{i-1}^j X_{i-1}^k] - \mathbb{E}[X_{i-1}^j]\mathbb{E}[X_{i-1}^k] + \rho_i \sigma_{\varepsilon,i}\sigma_{\eta,i}}{\sigma_{X_i^j}\sigma_{X_i^k}} =$$

$$\frac{\mathbb{E}[X_{i-1}^j X_{i-1}^k] - \mathbb{E}[X_{i-1}^j]\mathbb{E}[X_{i-1}^k]}{\sigma_{X_{i-1}^j}\sigma_{X_{i-1}^k}} \frac{\sigma_{X_{i-1}^j}\sigma_{X_{i-1}^k}}{\sigma_{X_i^j}\sigma_{X_i^k}} + \frac{\rho_i \sigma_{\varepsilon,i}\sigma_{\eta,i}}{\sigma_{X_i^j}\sigma_{X_i^k}} = \rho_{X_{i-1}^j, X_{i-1}^k} K_i + H_i,$$

where $K_i = \frac{\sigma_{X_{i-1}^j}\sigma_{X_{i-1}^k}}{\sigma_{X_i^j}\sigma_{X_i^k}}$ and $H_i = \frac{\rho_i \sigma_{\varepsilon,i}\sigma_{\eta,i}}{\sigma_{X_i^j}\sigma_{X_i^k}}$.

Remark 8.1. ***Correlation swaps***. *The recursion above can be used to enforce no-arbitrage relationships among correlation swaps of increasing maturities. Notice that the iteration is a function of the dependence structure of increments only. In this way, it provides a shortcut with respect to a more complex numerical computation of the dependence structure among levels at each point in time from the copula recursion.*

9 The Term Structure of Multivariate Equity Derivatives

In this section we show how to obtain the term structure of prices of multivariate derivatives in a market model with temporally independent, or conditionally independent increments with assigned spatial dependence structure of the increments. In order to stress the way in which the dependence of increments cumulate into that of the variables we stick to the vector independent model. We will address the main kinds of products, ranging from multivariate digital derivatives, to rainbow options and basket/spread options.

As for the recovery of information, notice that an important implication of our approach is that we can obtain arbitrage free prices by mixing calibration of the marginal distributions from options data and estimation of the cross-section dependence structure of the increments from historical time series. More explicitly, the procedure calls for:

- estimation of the dependence structure of increments from time series
- fitting of the smile for each asset involved
- simulation of the dependence structure of prices at different time horizons.

For the sake of illustration, we report here the pricing surfaces of the main multivariate products, under two different scenarios concerning the dependence structure of return innovations. For each product, we also report the multivariate term structure of prices consistent with the no-arbitrage requirement. Of course, the pricing surface is not unique, since the market is incomplete, but it provides an important reference point for pricing under the realized dependence.

For the sake of simplicity, in our simulation design we stick to the bivariate case and to the hypothesis that $C_i(u,v)$ does not depend on i, that is: $C_i(u,v) = C(u,v)$. Based on these assumptions, we obtain two simulated trajectories $(X_i^j)_{0\leq i\leq n}$ and $(X_i^k)_{0\leq i\leq n}$ using standard Monte Carlo simulation techniques for copula functions. The marginals are assumed to be Gaussian, and in order to ensure that the prices $e^{X_i^j}$ and $e^{X_i^k}$ be martingales, we assume that

$$\varepsilon_i \sim \mathcal{N}(-\frac{\sigma_\varepsilon^2}{2}, \sigma_\varepsilon^2)$$

and

$$\eta_i \sim \mathcal{N}(-\frac{\sigma_\eta^2}{2}, \sigma_\eta^2).$$

for $i = 0, ..., n$. In fact, in this case (and analogously for $e^{X_i^k}$)

$$\mathbb{E}(e^{X_i^j}|\mathcal{F}_{i-1}) = \mathbb{E}(e^{X_{i-1}^j+\varepsilon_i}|\mathcal{F}_{i-1}) = e^{X_{i-1}^j}\mathbb{E}(e^{\varepsilon_i}) = e^{X_{i-1}^j},$$

since $\mathbb{E}(e^{\varepsilon_i}) = e^{-\frac{\sigma_\varepsilon^2}{2}+\frac{\sigma_\varepsilon^2}{2}} = 1$.

As for the dependence structure of the increments, in the base scenario, return innovations in the two markets are assumed to be independent. In the alternative scenario, they are assumed to be dependent, and the dependence structure is assumed to be represented by a Clayton copula with 40% Kendall τ dependence.

Before getting into the details of each product, we report in Figure 1 the main innovation of our approach with respect to the copula technique that is currently in use. In standard applications, one would have estimated the dependence structure between the *levels* of the two assets, and, given the marginals, obtained the price. In our approach one would have estimated the dependent structure between the *increments* of the assets, and then, assuming independent increments, would have recovered the dependence structure by simulation. Only the latter approach provides arbitrage-free prices. Figure 1 describes the difference of the probability distribution obtained under the two approaches. In other terms, the Figure depicts the difference in the pricing kernel between a price made using copulas on the levels and an arbitrage-free copula.

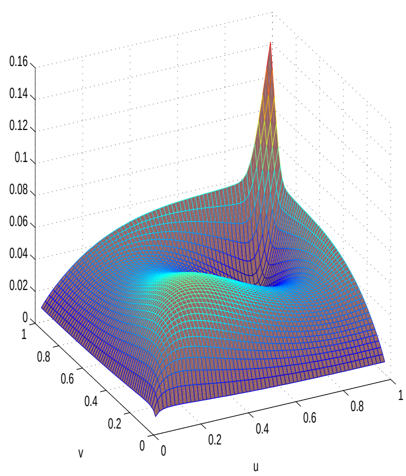

Figure 1: Pricing kernel difference between Clayton copula with 40% Kendall τ and the copula functions obtained with price increments linked with a Clayton copula with 40% Kendall τ.

9.1 Altiplanos

We start the analysis with multivariate digital products, called Altiplanos: these products give a fixed amount payoff if the prices of all assets in a basket are affected by some event (say none of them are below a given threshold on a given date). Pricing Altiplanos was the first, straightforward application of copulas to equity derivatives. In fact, due to the digital payoff, their price amounts to the computation of a copula function. However, since in Altiplano notes multivariate digital derivatives are used to determine the coupon pay-offs on a stream of dates $\{t_1, t_2, ..., t_n\}$, the pricing of this kind of securities quite naturally calls for temporal consistency of prices of the multivariate derivatives across different maturities. This temporal consistency feature is even more evident in *barrier Altiplanos*, in which the coupon is paid if all the assets are above a given threshold at a set of fixing dates, rather than on the payment date. Differently from the research surveyed above, however, we focus here on such temporal consistency.

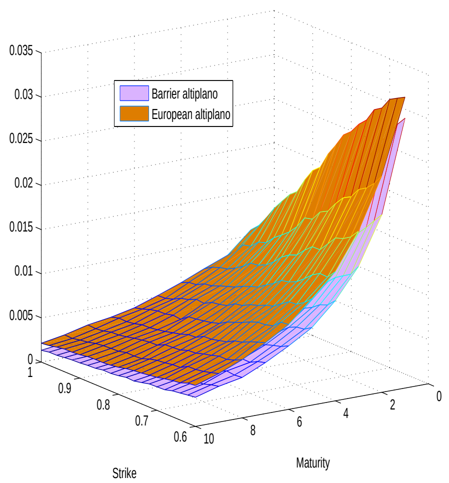

Figure 2: Arbitrage free pricing surface. European vs barrier Altiplanos. Clayton copula with Kendall τ equal to 40% .

In Figure 3 and 4 we report the term structure of the prices of European Altiplanos with same contract features and underlying assets but different dependence structure. Apart from Monte Carlo errors, the whole term structure appears to be shifted up (down) by a parallel amount by an increase (decrease) in the dependence structure.

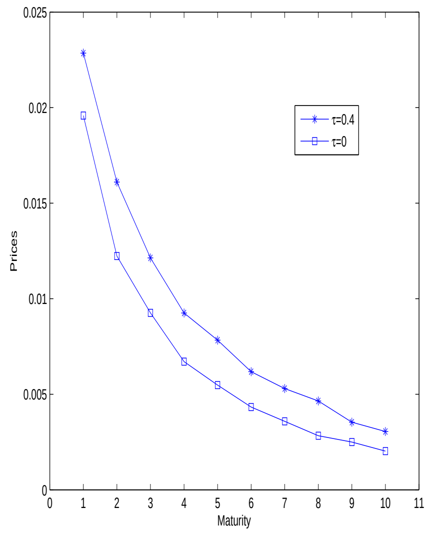

Figure 3: Arbitrage free term structure of European Altiplanos. Product copula vs Clayton copula with Kendall τ equal to 40% .

9.2 Everest

We now extends the analysis to rainbow products. In multivariate equity notes, call on the minimum of assets is typically used to provide the coupon payments over a guaranteed minimum return. These products are known as Everest. The coupon is paid according to the rule

$$\max(\min(S_T^1/S_0^1, S_T^2/S_0^2, ...S_T^m/S_0^m), 1+k)$$

where k is the guaranteed return. This is obviously equivalent to a bond paying a coupon equal to k percent for sure, plus a call on the minimum return to the assets, with strike equal to $1+k$

$$\max(\min(S_T^1/S_0^1, S_T^2/S_0^2, ...S_T^m/S_0^m) - (1+k), 0)$$

The product is actually the integral of multivariate digital options. In Figure 4 we report the term structure of call on the minimum options. The term

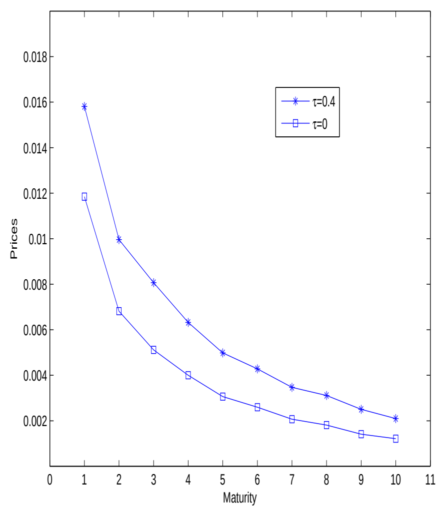

Figure 4: Arbitrage free term structure of barrier Altiplanos. Product copula vs Clayton copula with Kendall τ equal to 40% .

structure for this product is not monotone in general. Moreover, not only the product is long correlation as expected, but the term structure is also made steeper by an increase in correlation.

9.3 Spread options

The last class of multivariate product we address is spread options. The payoff is

$$\max(S_T^1/S_0^1 - S_T^2/S_0 - K, 0)$$

Notice that this is nothing but a specific instance of the basket option, that is an option on the sum of two assets, in the case in which the second asset is taken with the negative sign. To this purpose, it is worth reminding that there is a specific relationship between the copula function to be used between two variables and monotonic transformations of them. Namely, in the case at hand, if C is the copula linking S_1 and S_2, the copula linking S_1 and $-S_2$ (and any other strictly decreasing transformation of S_2) has to be

$$\widetilde{C} \equiv u - C(u, 1 - v)$$

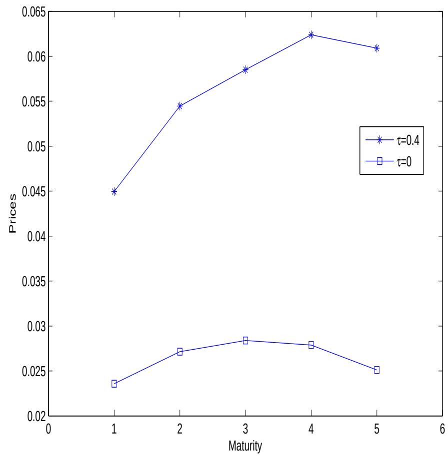

Figure 5: Arbitrage free term structure of call on the min options. Product copula vs Clayton copula with Kendall τ equal to 40% .

Notice that the product obviously turns out to be short correlation. In Figure 6 we also report the sensitivity of the whole term structure to changes in correlation. Opposite to what happens for options that are long correlation, the effect of an increase of correlation is to flatten out the term structure of the product.

10 Conclusions

In this paper we provide a model in which multivariate equity derivatives are priced with copulas in an arbitrage-free setting. The approach consists of a discrete time market model in which increments of log-prices are linked across assets by copula functions, in the spirit of the SCOMDY models proposed in the recent econometric literature of time series. In this model, we show that martingale prices can be recovered by first specifying the dependence structure of log-increments and then linking them with the proper copula functions. This approach shows that copulas cannot be freely chosen and applied to the dependence structure of prices of different maturities, since the martingale restriction imposes constraints across maturities. Moreover, only in very few cases can these copula functions be obtained in closed form, and in general they must be

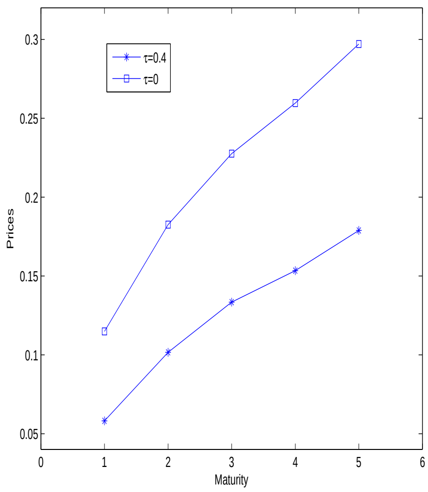

Figure 6: Arbitrage free term structure of spread options. Product copula vs Clayton copula with Kendall τ equal to 40% .

simulated. Computation can be hugely simplified if the increments are assumed to be independent. Even though the assumption of independent increments is often criticized on the grounds that makes difficult to explain the skew observed in the market, in most models the independent increment property can be recovered under a suitable time change.
As examples, we apply the procedure to the evaluation of term structures of prices of standard multivariate equity derivatives. We find that and increase in the dependence of increments across all maturities brings about a parallel upward shift for digital products, it increases the slope for call on min options, while it flattens out the term structure of spread options.

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