# Multivariate Digital Options with Memory

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**Abstract**

We study a class of multivariate digital products called Altiplanos. These products may be structured according to two general features: i) they may be univariate or multivariate; ii) they may be European or with barrier. In addition to that, they may be endowed with exotic characteristics. One of these is the so-called "memory" feature, which prescribes that the first time when the underlying event takes place, coupons are paid for all the previous periods in which it had not occurred. The task of this paper is to provide new results for the evaluation of this clause. We show that the memory feature provides that a digital option paying all coupons in the final date plays a dominant role in the evaluation. Concerning sensitivity, the value of digital products with memory are positively sensitive to an increase in cross-section correlation and to a decrease in temporal correlation.

Keywords: copula functions, Markov processes, multivariate options, memory feature, correlation products

## 1 Introduction

The recent developments of the market for structured finance products have witnessed the increasing use of multivariate options to provide diversification to investors and to allow financial intermediaries to exploit changes in the dependence structure of the risk factors. For these reasons, these structures are called correlation products. The first structured financial products enabling to exploit changes in dependence were launched in the credit market quite a long time ago, and have reached a large diffusion, as well as a continuous process of standardization: nowadays, traders may invest in very liquid instruments to bet on changes in correlation among the main names in the credit default swap market. More recently, multivariate products have been proposed in equity-linked structured finance as well. A typical and basic family of products is represented by digital structures paying a unit of cash if the price of a set of assets is above a given barrier. Products in this family are called Altiplano notes, and are widely used not only in structured finance but also in life insurance policies. In its simplest form, an Altiplano note pays digital coupons, and each one of them pays a fixed sum if at some future date the price of each and every asset in a basket of securities is above a corresponding strike. It is quite easy to check that this product is long correlation, meaning that the value of the embedded options increases with the increase in correlation. Alternatively, from the viewpoint of risk, one could restate this saying that the choice of a well diversified basket of assets effectively curbs the risk of the product. Then, the main risk factor of this product is then the cross-section dependence structure of a set equity prices at a given future date. For this reason, we may more precisely define this product a European Altiplano.

In market practice, more complex clauses are typically included in these products. One is that the fixed amount is paid if the prices of the assets remain above the barrier across a whole set of fixing dates: the typical structure is that the coupon is paid at the end of each year if all the assets are above the corresponding barriers at the end of each month in that year. We call this product a "barrier Altiplano". Another typical, albeit less usual, clause is the so called "memory" feature: the first time that a coupon is paid, all of the previous coupons that had earned zero are also giving the payoff. These clauses make the evaluation of the product more complex and the risk analysis on correlation sensitivity more involved. The "barrier" feature has been already investigated in Cherubini and Romagnoli (2009a). In this paper, it is our objective to complete the analysis focussing on the impact of the "memory" feature in the same framework. We show that the same "bootstrap technology" applied to barrier Altiplanos can be applied in the backward direction to cover this important clause that is very often included in European contracts.

The plan of the paper is as follows. In Section 2 we define Altiplanos and briefly go over the most relevant literature on the subject. In section 3 we describe the market model used for pricing. In section 4 we derive the main result of the paper, which is the price of a stream of digital coupons with the memory feature. In section 4 and 5 we specify the pricing formulas for univariate and multivariate cases respectively, and we present a sensitivity analysis comparing products with the memory feature against European and barrier products. Section 6 concludes.

## 2 Altiplanos

Altiplanos are digital derivatives included in structured products and life insurance policies, and are typically written on equity underlying assets. By a very general taxonomy, the structure may be thought to be of four kinds:

- European univariate Altiplano: this is a standard digital option.
- Barrier univariate Altiplano: this is a discrete monitoring barrier option
- European multivariate Altiplano: this is a multivariate digital option
- Barrier multivariate Altiplano: this is a discrete monitoring barrier multivariate option.

The most typical products are multivariate European and barrier Altiplanos. When buying or selling these products one wants to place a bet on three risk factors: i) directional movements of the underlying assets, ii) changes in the volatility of the underlying assets; iii) changes in correlation among the underlying assets. The product is clearly long in the direction of the underlying asset and (less clearly) short with respect to changes in volatility. As for correlation, the typical sign is positive even though the kind of response of the product on the correlation may be altered by special structuring choices.

Since arbitrage-free pricing requires that the price of multivariate digital options has to be the discounted value of a joint probability distribution, it is natural to use copula functions as the ideal tool to compute the price. Since copula functions enable to write joint distributions as a function of univariate marginal distributions, it allows to recover the prices of multivariate digital options that are consistent with the univariate digital options that can be calibrated from option quotes (see Nelsen, 2006 for a broad mathematical introduction and Embrechts et al, 2003 and Cherubini et al, 2004 for applications in finance). The first attempt to evaluate multivariate options consistently with the univariate ones was done by Rosenberg (1998). Cherubini and Luciano (2002) extended the approach using Archimedean copulas. In Cherubini et al (2004), chapter 8, the results are extended to provide a general derivation of put-call parity relationships in a bivariate setting. Cherubini and Romagnoli (2009b) provided an algorithm to compute put-call parity in a multivariate setting of up to 20 underlying assets.

In all these works, copula functions are used to recover the price of multivariate European digital claims by a static replication argument applied to univariate digital options. A natural question arises concerning the dynamics of these prices as new information flows to the market. News may refer not only to marginal distributions but also to the dependence structure. There is also evidence that these shocks may not be independent, with the dependence among the underlying assets increasing when prices in the market are falling (Longin and Solnik, 2001 and Ang and Chen, 2002). For this reason a recent stream of literature introduced *dynamic copulas*. Namely, Patton (2006) introduced the concept of *conditional copula*, in a paper that had been circulating since 2001. Fermanian and Webkamp (2004) proposed an alternative approach defining the new concept of *pseudo-copulas*. Van den Goorbergh, Genest and Werkerc (2005) proposed a different approach in which the dynamics of the dependence structure was determined as a function of the path of volatilities.

The extension of pricing models to include a dynamics of dependence should not be confused with another problem, which has to do with more complicated multivariate digital products that are bought and sold in the market. A typical and usual extension is to substitute the path of the underlying assets to the level of the prices of assets at expiration. A standard way to make the product path-dependent is to resort to "barrier Altiplanos", so that the coupon is paid if each and every asset remains above the corresponding barrier for all the fixing dates defined in the contract. A second, not necessarily alternative, clause that may make the product path dependent is the so called "memory feature" according to which each time the coupon is paid, it is also paid for the previous periods in which it had turned out to be zero.

Path-dependent versions of multivariate options quite naturally raise the problem of temporal dependence, on top of the usual cross-section concept. Surprisingly enough, the temporal dependence problem has been overlooked by pricers and risk managers. While it is quite standard to care about product sensitivity to changes in cross-section dependence among the prices of the different assets, and about long and short correlation positions, it is more unusual to ear about sensitivity to changes in temporal correlation across the assets. Actually, if one buys a "barrier Altiplano" he should keep an eye on both cross-section and temporal dependence as the main risk factors, and the same should be true for Altiplanos with "memory", as we will show below. In fact, one could think of a product paying a digital pay-off if a single asset remains above a barrier on a set of fixing dates: the price of this univariate barrier Altiplano, which is nothing but a digital barrier option (a "no-touch" contract) is sensitive to changes in temporal dependence only, just like the European Altiplano is only sensitive to cross-section dependence.

In principle, the copula approach that is used to represent cross-section dependence and to price European Altiplanos could be also applied to study temporal dependence. It is quite surprising that while copulas have been used to study non linear temporal dependence in econometrics, the same has not been done in option pricing applications. The application of copula functions to represent the dynamics of a stochastic process goes back to the idea that Markov processes can be represented as a sequence of products of copula functions, where the product operator was defined in Darsow, Nguyen and Olsen (1992) (see Ibragimov, 2005 for the multivariate extension). Recently, this idea was applied by Cherubini and Romagnoli (2009a) to recover the price of "barrier Altiplanos". In this paper we show how to address the problem of the presence of a "memory feature" under the same setting.

# 3 The Market Model

Here we briefly review the model proposed in Cherubini and Romagnoli (2009a) focussing on the features that will be exploited to price products with "memory". We assume a filtered probability space $\{\mathbf{\Omega}, \Im_t, Q\}$ satisfying the usual conditions with $Q$ the risk-neutral measure. We consider a set of assets $\{S_1, S_2, ..., S_m\}$ and a set of dates $\{t_1, t_2, ..., t_n\}$.

The market model is based on three assumptions:

1. *Risk Neutral Marginal Distributions.* All assets are martingale processes with respect to their natural filtration.

2. *Markov Property.* The prices of assets are Markov processes, as required by the efficient market hypothesis

3. *No-Granger Causality.* Information about past and current dynamics of other assets has no effect on the future value of one asset, over and above the filtration generated by the asset itself.

Notice that the *No-Granger causality* condition actually allows for extension of the martingale property from the filtration generated by each asset to the overall filtration generated by all assets.

The requirement of asset prices being Markov processes allows to apply to each asset the copula representation proposed by Darsow, Nguyen and Olsen (1992), that we report here for the ease of the reader.

**Theorem 1** *A real valued stochastic process $S(t_j)$ is a Markov process if and only if, for all positive integers $n$ and all $t_1 < t_2 < \ldots < t_n$*

$$C_{S(t_1)\ldots S(t_n)} = C_{S(t_1),S(t_2)} \star C_{S(t_2),S(t_3)} \star .. \star C_{S(t_{n-1}),S(t_n)}$$

*where $C_{S(t_1)\ldots S(t_n)}$ is the copula of $(S(t_1), \ldots, S(t_n))$, and $C_{S(t_{k-1}),S(t_k)}$ is the copula of $(S(t_{k-1}), S(t_k))$.*

For the specific form of the $\star$-product (star-product) operator we refer the reader to the original paper or to the final chapter of Nelsen (2006). We only stress here that this is merely a way of writing the Chapman-Kolmogorov equation in the language of copulas, exploiting the fact that conditional distributions are actually the partial derivatives of the copula function.

Notice that this copula representation of a Markov process immediately provides a general representation of the running maxima and minima of the asset price on the set of dates $\{t_1, t_2, ..., t_n\}$. This is directly derived from the definition of running maxima and minima.

**Definition 2** *The running minimum and maximum of each asset $S$ starting from time $t_r$ up to time $t_j$, $r < j$ are defined as*

$$\begin{aligned} M(t_r, t_j) &\equiv \max\{S(t_k); t_r \leq t_k \leq t_j\} \\ m(t_r, t_j) &\equiv \min\{S(t_k); t_r \leq t_k \leq t_j\} \end{aligned}$$

*and the corresponding probability distributions are given by*

$$\begin{aligned} \Pr(M(t_r, t_j) \leq B) &= C_{S(t_r)\ldots S(t_j)}\left(Q^{t_r}(B), ..., Q^{t_j}(B)\right) \\ \Pr(m(t_r, t_j) > B) &= \overline{C}_{S(t_r)\ldots S(t_j)}\left(\overline{Q}^{t_r}(B), ..., \overline{Q}^{t_j}(B)\right) \end{aligned}$$

*where $Q^{t_j}(B) \equiv \Pr(S(t_j) \leq B)$ and $\overline{Q}^{t_j}(B) \equiv \Pr(S(t_j) > B)$. Likewise, $\overline{C}$ is the survival copula of $C$.*

While we refer to the literature recalled above for the properties of the star-product operator, we focus on one which is actually paramount for the application discussed here. This property is associativity, meaning that:

$$\begin{aligned} C_{S(t_1),S(t_2)} \star C_{S(t_2),S(t_3)} \star C_{S(t_3),S(t_4)} &= \left(C_{S(t_1),S(t_2)} \star C_{S(t_2),S(t_3)}\right) \star C_{S(t_3),S(t_4)} \\ &= C_{S(t_1),S(t_2)} \star \left(C_{S(t_2),S(t_3)} \star C_{S(t_3),S(t_4)}\right) \end{aligned}$$

The relevance of this result for our application emerges in full clarity if we apply this property to the running maximum (or minimum). In fact, from the definition of running maximum the associativity relationship above implies

$$\Pr(M(t_1, t_4)) = \Pr(M(t_1, t_3)) \star C_{S(t_3),S(t_4)} = C_{S(t_1),S(t_2)} \star \Pr(M(t_2, t_4))$$

We then have recursion relationships between the running maxima starting and ending at different times. Recursion relationships like these are typically called *bootstrap* in the financial jargon.

**Definition 3** ***Bootstrap for running maxima and minima****. Given a Markov process defined in discrete time, the running maxima can be computed by the recursion*

$$\Pr(M(t_1, t_n)) = \Pr(M(t_1, t_{n-1})) \star C_{S(t_{n-1}),S(t_n)}$$

*which is defined* ***forward bootstrap*** *or alternatively the recursion*

$$\Pr(M(t_1, t_n)) = C_{S(t_1),S(t_2)} \star \Pr(M(t_2, t_n))$$

*which is defined* ***backward bootstrap****. The same result hold for running minima.*

This particular concept of *bootstrap* was first introduced in Cherubini and Romagnoli (2009a) for application to barrier Altiplanos. In that paper it was only defined what here we call the *forward bootstrap* because it was the only one needed to price barrier Altiplanos. In this paper the definition is extended to encompass the *backward bootstrap.* While the forward boostrap links barrier Altiplanos *ending* at consecutive maturities, the backward bootstrap links barrier Altiplanos *starting* at consecutive dates. As it will be clear in the next paragraph, backward boostrap is instead needed in the pricing of products with the memory feature.

Notice that this result becomes even more easy to handle if we further restrict the analysis to the subset of copulas which are closed under the star-product operators. This essentially means that the copula function is not changed after the star-product operator is applied. The bad news is that copula functions which are endowed with this property are not many (see Nelsen, 2006, for examples). The good news is that one such copula is the Brownian one, defined as

$$C(u,v)=\int_0^u \Phi\left(\frac{\sqrt{t}\Phi^{-1}(v)-\sqrt{s}\Phi^{-1}(w)}{\sqrt{t-s}}\right)dw \tag{1}$$

where $u = \Pr(S(t) \leq B_t)$, $v = \Pr(S(s) \leq B_s)$ for $s < t$ and $\Phi(.)$ is the standard normal distribution.

Notice that this is not necessarily restrictive since a very well known result due to Monroe (1978) states that all semi-martingale processes can be written as Brownian motion processes in which the scale of time is suitably changed. This is the stochastic clock approach widely used in finance (see, for all, Carr and Wu, 2004 and references therein). In the copula function language, this change of time is represented as

$$C(u,v;\omega)=\int_0^u \Phi\left(\frac{\sqrt{h(t,\omega)}\Phi^{-1}(v)-\sqrt{h(s,\omega)}\Phi^{-1}(w)}{\sqrt{h(t,\omega)-h(s,\omega)}}\right)dw \tag{2}$$

where as previously $u = \Pr(S(t) \leq B_t)$, $v = \Pr(S(s) \leq B_s)$ for $s < t$ and $h(t,\omega)$ is a suitable increasing stochastic function. So, limiting the analysis to the time changed Brownian copula is not to be considered restrictive in any way.

# 4 The memory feature

We now apply the framework above to the problem of evaluating a digital product, that is an Altiplano, with "memory". We recall that this clause states that the first time that the event takes place the coupons are also paid for all previous periods for which it had not occurred. The key to pricing this product is that if the event takes place at time $t_j$ and does not occur any more afterwards, then the number of coupons paid is $j$. By the amount $c(j)$ we denote the overall amount of coupons paid in that case, where $c(j)$ is a suitable function considering the coupons established in the contract and the possible sequences of payments up to time $t_j$. As the simplest example, if the coupon of each period is constant and the risk-free rate is assumed to be equal to zero, we may write $c(j) = j\bar{c}$, where $\bar{c}$ is the constant coupon.

We now formalize the pricing approach. We start by defining $\mathcal{A}_j$ the characteristic function denoting that the event takes place at time $t_j$. The term $\overline{\mathcal{A}}_j$ denotes the characteristic function of the complement. Assume the set of fixing dates is $\{t_1, t_2, ..., t_n\}$. We first observe that the product will pay coupons $c(n)$ if and only if the event takes place at time $t_n$, that is if $\mathcal{A}_n = 1$. If we denote $\mathbf{Q}(\mathcal{A}_n)$ the risk-neutral probability of the event taking place at time $t_n$, and $v(t_0, t_n)$ the discount function, then the price of the coupon stream will be $v(t_0, t_n)c(n)\mathbf{Q}(\mathcal{A}_n)$.

Let us now move back to the case $\mathcal{A}_{n-1} \cap \overline{\mathcal{A}}_n$. In this scenario the event takes place at time $t_{n-1}$ but does not takes place at time $t_n$, so that the product pays a coupon stream of $c(n-1)$. Notice that the corresponding risk-neutral probability can be written as

$$\mathbf{Q}(\mathcal{A}_{n-1} \cap \overline{\mathcal{A}}_n) = \mathbf{Q}(\overline{\mathcal{A}}_n) - \mathbf{Q}(\overline{\mathcal{A}}_{n-1} \cap \overline{\mathcal{A}}_n)$$

Extending the argument by induction we have that the risk-neutral probability of a scenario of a coupon stream equal to $c(n-j)$ is given by

$$\mathbf{Q}(\mathcal{A}_{n-j} \cap \overline{\mathcal{A}}_{n-j+1} ... \cap \overline{\mathcal{A}}_n) = \mathbf{Q}(\overline{\mathcal{A}}_{n-j+1} \ldots \cap \overline{\mathcal{A}}_n) - \mathbf{Q}(\overline{\mathcal{A}}_{n-j} \cap \overline{\mathcal{A}}_{n-j+1} ... \cap \overline{\mathcal{A}}_n)$$

with $j = 1, 2, \ldots n-1$.

We are now in a position to state the main result of this paper:

**Proposition 4** *Assume a digital product with memory with fixing dates* $\{t_1, t_2, \ldots, t_n\}$ *and a set characteristic functions* $\mathcal{A}_j$*, spotting the event that triggers the payoff at any time* $t_j$*. The price of this product is given by*

$$\begin{aligned} A(t_0, t_n) &= v(t_0, t_n)\mathbf{Q}(\mathcal{A}_n)c(n) + \\ &+ \sum_{j=1}^{n-1} v(t_0, t_{n-j}) \left( \mathbf{Q}(\overline{\mathcal{A}}_{n-j+1} \ldots \cap \overline{\mathcal{A}}_n) - \mathbf{Q}(\overline{\mathcal{A}}_{n-j} \cap \overline{\mathcal{A}}_{n-j+1} ... \cap \overline{\mathcal{A}}_n) \right) c(n-j) \end{aligned} \quad (3)$$

An important remark immediately emerges from a first look at the pricing formula, and will be confirmed in the illustrative examples below. In the value of this product, a dominant role is played by the probability that the trigger event occurs in the last date. In fact, this is the first term in equation (3); it can be noticed that this is the only term which is not damped by one of opposite sign, and for this reason it turns out to be dominant. This endows the contract with a digital-like feature of a European product that pays all the coupons at the final date.

# 5 Univariate Altiplanos with Memory

In the case of a single underlying asset, the trigger event $\mathcal{A}_j$ is simply defined as

$$\mathcal{A}_j \equiv \mathbf{1}_{\{S(t_j)>B\}}$$

and the complementary event is obvious. The discounted risk-neutral probability of earning $n$ coupons is then simply $v(t_0, t_n)Q(S(t_n) > B)$, that is the price of a European digital option expiring at time $t_n$. Furthermore, using the definition of running maximum it is easy to check that the risk-neutral probability of earning $n - j$ coupons is instead

$$\mathbf{Q}(M(t_{n-j+1}, t_n) \leq B) - \mathbf{Q}(M(t_{n-j}, t_n) \leq B)$$

From this equation it turns out now clear that the pricing of the memory feature actually requires the application of a *backward bootstrap* algorithm.

In order to illustrate the behavior of a univariate digital product with memory we assumed an underlying asset with marginal log-normal distribution with 30% volatility in the base scenario. Temporal dependence is represented by a time-changed Brownian copula with Ornstein-Uhlembeck clock as in Cherubini and Romagnoli (2009a). In the base scenario the dynamics is instead given by the standard Brownian copula.

Figure 1 reports the relationship between the value of the coupon stream and different levels of volatility of the marginal distribution, against the base scenario with 30% volatility. For comparison, the schedule representing the digital product with memory is reported along with those of an European product and a barrier product, that is a *no-touch* option. We notice that all instruments are short volatility, and that the digital product with memory is the most valuable.

INSERT FIG.1 ABOUT HERE

Figure 2 reports the relationship between the value of the coupon stream and different barrier levels in the base scenario with 30% volatility. We notice that all instruments, i.e. the digital product with memory, the European product and the barrier product, are decreasing with respect to the barrier level, and that the digital product with memory is also in this case, the most valuable.

INSERT FIG.2 ABOUT HERE

Figure 3 shows how the values of the products are affected if the mean-reversion parameter in the Ornstein-Uhlembeck clock is increased. We remind that as the mean reversion increases, temporal dependence decreases down towards independence. The comparison of the barrier option and the digital option with memory shows that the impact is opposite. While the curve is definitely decreasing for the barrier product it is slightly increasing for the product with memory.

INSERT FIG.3 ABOUT HERE

## 6 Multivariate Altiplanos with Memory

We now extend the analysis to the case in which the underlying is not a single asset, but a basket of securities, which is the usual case in the market. In this case, assuming a basket of $m$ assets, the event can be generally written as

$$\mathcal{A}_j \equiv \mathbf{1}_{\{(S_1(t_j)>B_1)\cap...\cap(S_i(t_j)>B_i)\cap...\cap(S_m(t_j)>B_m)\}}$$

The analysis carries over just like in the univariate case. The only remarkable difference is that in this case the price cannot be written in terms of running maxima as in the univariate case. This is also different from the pricing result for multivariate barrier Altiplanos proved in Cherubini and Romagnoli (2009a), that is that the price can be written in terms of the distributions of the running minima as marginals. To understand the point, notice that the probability of the complementary event $\overline{\mathcal{A}}_j$ is

$$\begin{aligned}\mathbf{Q}(\overline{\mathcal{A}}_j) = 1 - \mathbf{Q}\left((S_1(t_j) > B_1) \cap ... \cap (S_i(t_j) > B_i) \cap ... \cap (S_m(t_j) > B_m)\right) \\ = 1 - \mathbf{C}(\mathbf{Q}(B_1), ..., \mathbf{Q}(B_m))\end{aligned}$$

and the function $1 - \mathbf{C}(u_1, ..., u_m)$ is called *co-copula*, corresponding to the **OR** operator (see Cherubini et al.,2004).

To illustrate, we consider a homogeneous basket of 20 assets with the same feature (lognormal distribution with 30% volatility) as the univariate case above. We assume a constant cross-section dependence corresponding to 40% rank correlation (Spearman's $\rho$). As before, the dynamics of each asset is described by a Brownian copula in the base scenario and a time changed Ornstein-Uhlembeck Brownian copula in scenarios with different temporal dependence.

Figure 4 and 5 reports the value of the Altiplano with memory for increasing volatility and increasing barrier level, as before in comparison with the European and the barrier Altiplanos. We notice that all instruments are short volatility and decreases when the barrier level increases, and that the product with memory is the most valuable.

INSERT FIG.4 ABOUT HERE

INSERT FIG.5 ABOUT HERE

Figure 6 and 7 reports the value of the Altiplano with memory for increasing number of assets and increasing cross-section correlation, as before in comparison with the European and the barrier Altiplanos. The value of the Altiplano with memory is again uniformly more valuable than the other two. Apart from the difference of value, the behavior is again similar across the different types of contract. The value of all of them decreases when the number of asset increases and the increases with cross-section dependence, confirming in all cases a gain from diversification.

INSERT FIG.6 ABOUT HERE

INSERT FIG.7 ABOUT HERE

In figure 8 we plot the value of the Altiplano with memory against that with barriers for increasing values of the mean reversion of the Ornstein-Uhlembeck stochastic clock, as we did in the univariate case. The difference is now quite marked. While the value of the barrier Altiplano is decreasing for increasing values of the stochastic clock, as already shown in Cherubini and Romagnoli (2009a), the opposite takes place for the Altiplano with memory. There is also a marked difference in the shape of the curve, which is slightly convex for the memory case and concave for the barrier product.

INSERT FIG.8 ABOUT HERE

To conclude, and to understand the rationale behind our results, in figure 9 and 10 we plot the values of the scenarios used in our pricing approach. The curve links the cumulative number of coupons paid to their discounted present value (mainly, the risk corresponding risk neutral probability). In figure 6 we show the impact of a change in cross-section dependence. Notice that the curve is shifted upwards and made more and more concave by and increase in correlation. Figure 10 reports the effect of a change in temporal dependence. An increase in the mean reversion parameter brings about a decrease of concavity. Namely, the probability of every scenario of $j$ coupons is shifted upwards. Remember that an increase in the mean reversion parameter corresponds to a decrease in temporal correlation. So, digital products with memory are proved to be short with respect to temporal correlation.

INSERT FIG.9 ABOUT HERE

INSERT FIG.10 ABOUT HERE

## 7 Concluding remarks

In this paper we addressed the problem of how the different structuring choices in the design of multivariate digital options may affect the value of the product, and its sensitivity to changes in the risk factors. Particular focus is devoted to the so-called "memory" feature. This feature requires that each time a coupon is paid, the payment of all the previous coupons is also granted. This choice is compared with that of a multivariate barrier feature, according to which each coupon is paid if all the assets in a basket remain above a given threshold for all the fixing dates in a set. As for sensitivity analysis, it is carried out in the market model developed in Cherubini and Romagnoli (2009a) in which copulas are employed to represent both cross-section and temporal dependence. The first result is that the memory feature increases the value of the coupon stream. The reason is that this feature has the effect of making the product similar to a digital option paying all coupons on the furthest date. We then carry out a sensitivity analysis of the product in comparison with barrier Altiplanos. It turns out that the two choices lead to opposite sensitivities of the products to changes in temporal dependence. So, in periods in which the market is more hectic, that is when the stochastic clock increases, the value of the product with memory is increased while that of the barrier Altiplano moves in the opposite directions. This result is another proof that these correlation products are not only sensitive to changes in the dependence across the assets, but also to changes in temporal dependence. More to the point, the case shown in this paper proves that for some structuring choices it is the latter concept of dependence that may make the difference. So, both barrier and the memory features are technique to make a multivariate digital option path-dependent, but while the former makes the product long in both cross-section and temporal dependence, the latter makes the digital product long in cross-section and short in temporal dependence.

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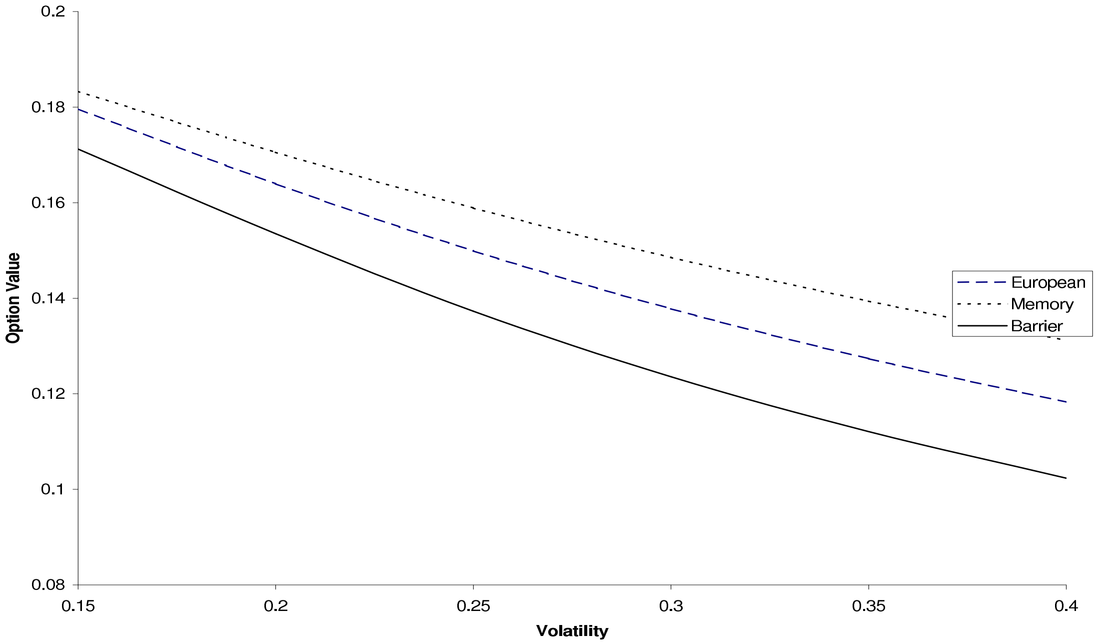


Figure 1: Univariate digital options, European, with barriers and with memory: value of coupons of a 5 year note, different volatilities.

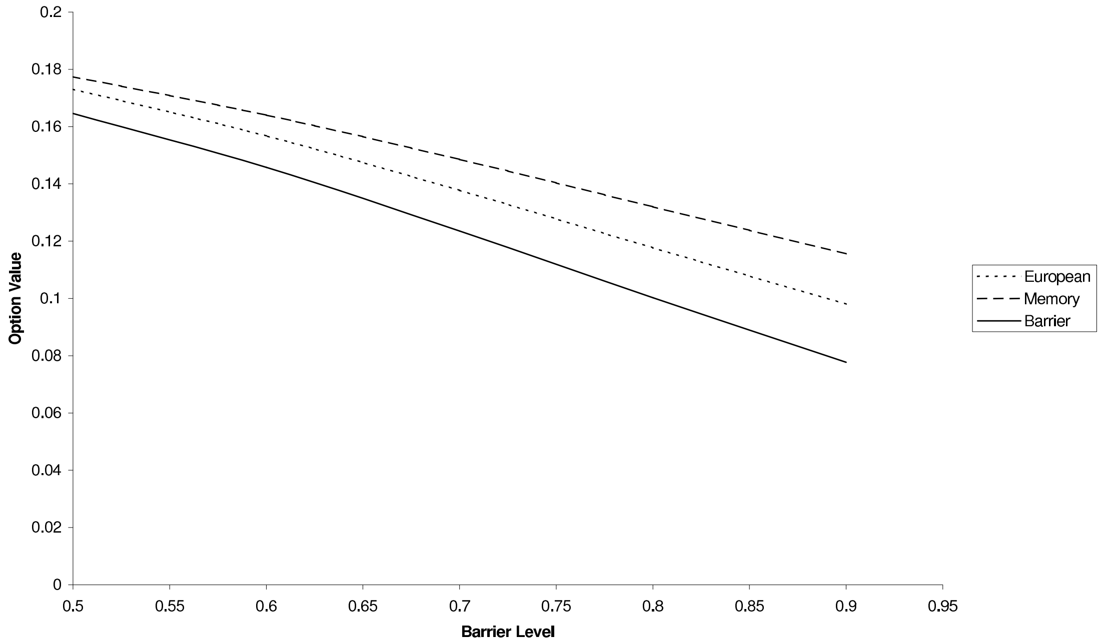


Figure 2: Univariate digital options, European, with barriers and with memory: value of coupons of a 5 year note, different barrier levels.

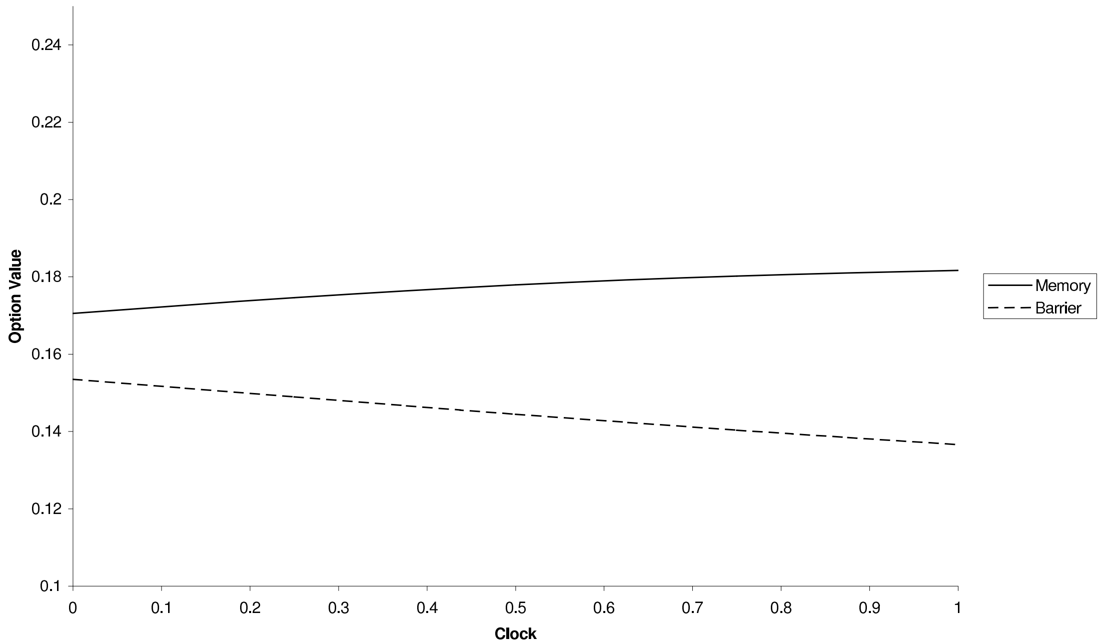


Figure 3: Univariate digital options, with barriers and with memory: different mean reversion parameters of the Ornstein-Uhlembeck clock.

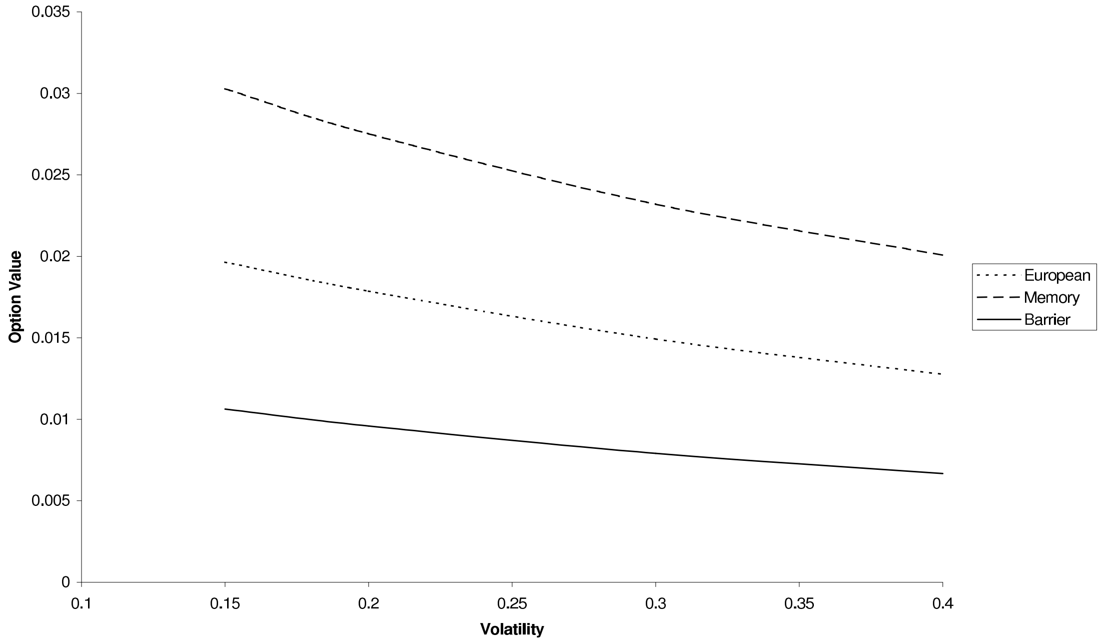


Figure 4: Multivariate digital options, European, with barriers and with memory: value of coupons of a 5 year note, different volatilities.

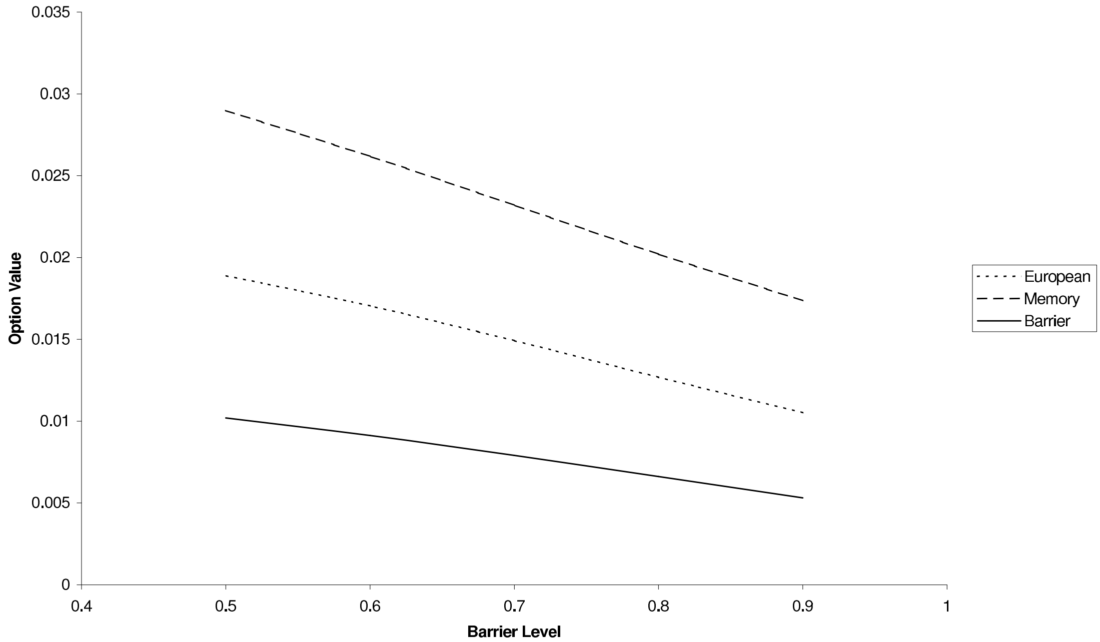


Figure 5: Multivariate digital options, European, with barriers and with memory: value of coupons of a 5 year note, different barrier levels.

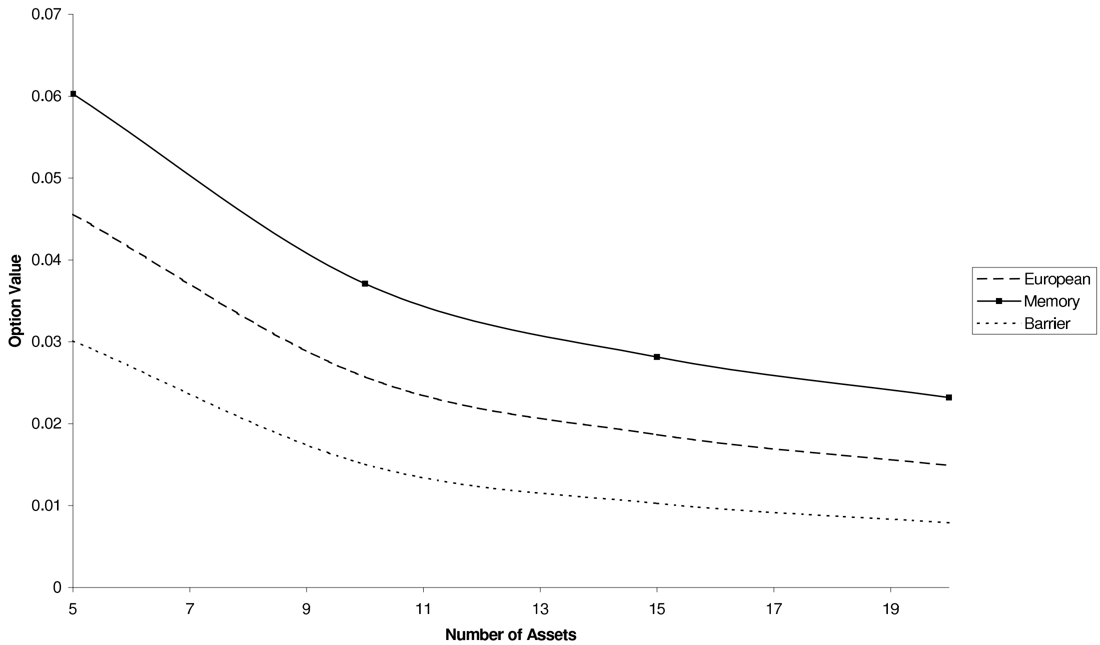


Figure 6: Multivariate digital options, European, with barriers and with memory: value of coupons of a 5 year note. Diversification with different number of assets.

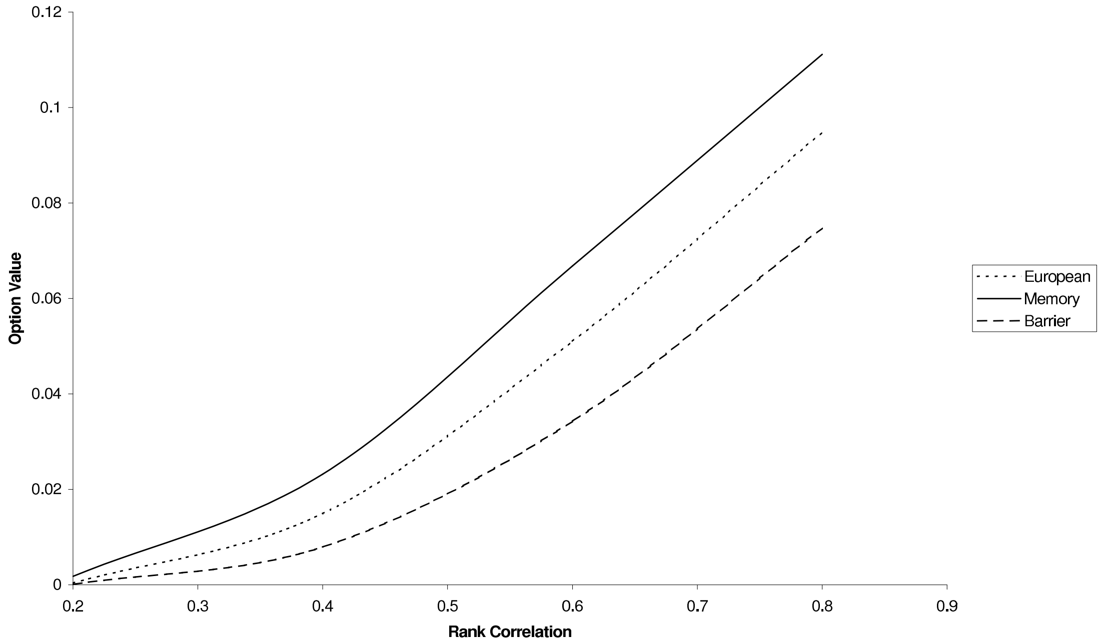


Figure 7: Multivariate digital options, European, with barriers and with memory. Increasing cross-section dependence parameters.

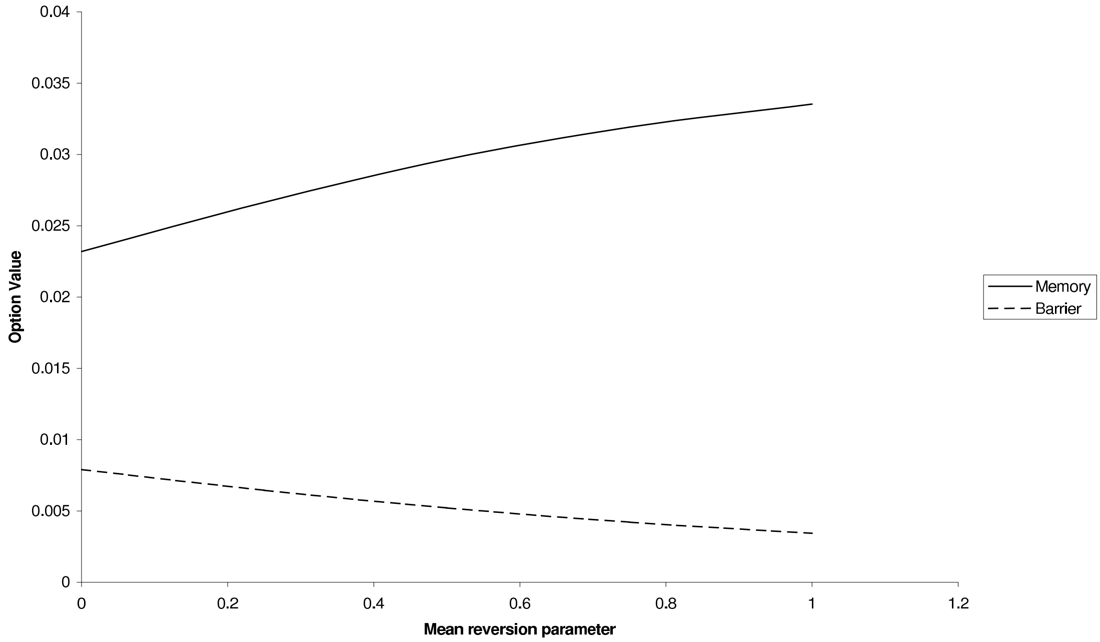


Figure 8: Multivariate digital options, with barriers and with memory: different mean reversion parameters of the Ornstein-Uhlembeck clock.

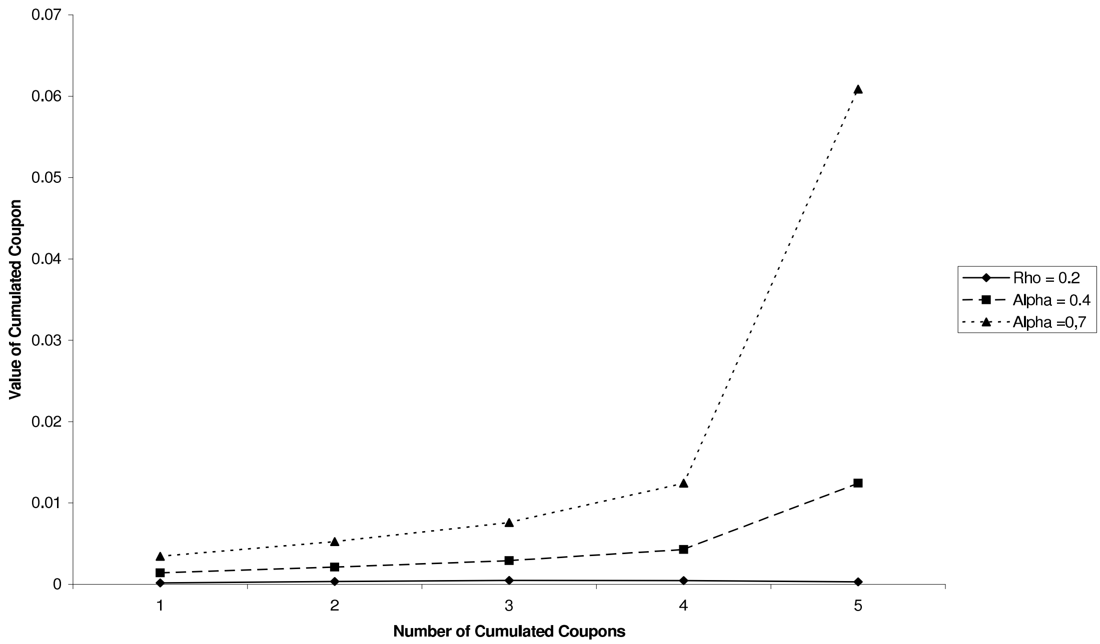


Figure 9: Cumulated coupons of a multivariate digital option with memory. Different cross-section dependence parameters.

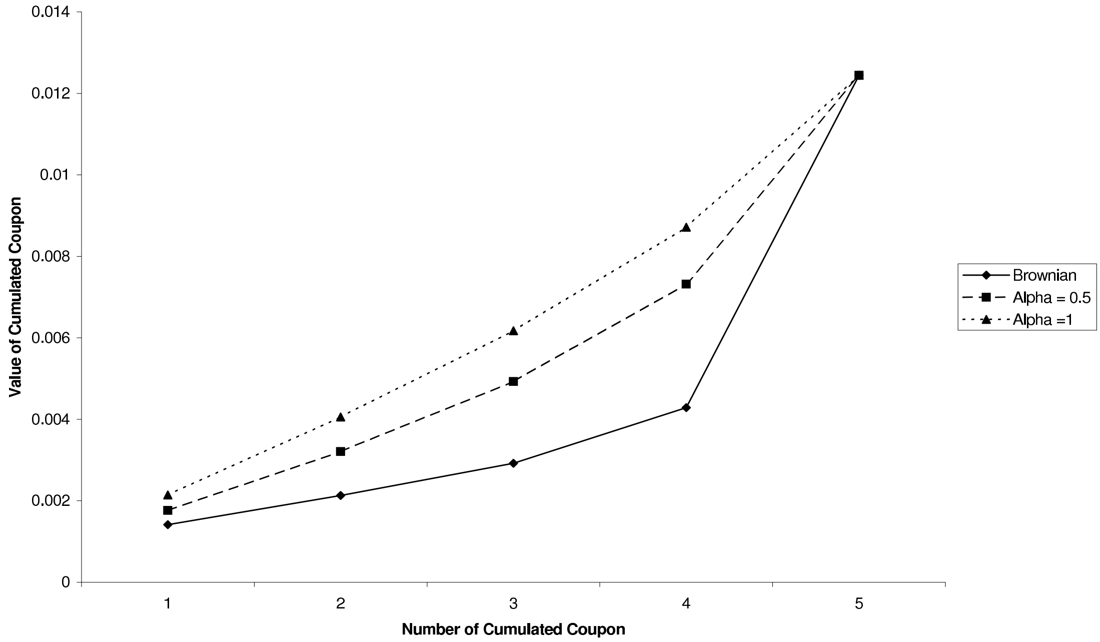


Figure 10: Cumulated coupons of a multivariate digital option with memory. Different mean reversion parameters of the Ornstein-Uhlembeck clock.